Problems for Lecture 15: Answers

1. First we rotate the <u>coordinate system</u> 45° anti-clockwise by applying the rotation matrix $\mathbf{R}_{-45^\circ} = \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{pmatrix}$ which is equivalent to rotating a vector clockwise by 45°, hence the negative sign! This transformation is followed by an extension by a factor 2 along the "new" x'-asis by applying the function $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Finally, we rotate the <u>coordinate system</u> 45° clockwise by applying the rotation matrix $\mathbf{R}_{45^\circ} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}$ which is equivalent to rotating a vector anti-clockwise by 45°. The composite transformation is the matrix product of these three matrices:

$$\begin{aligned}
\mathbf{R}_{45^\circ}\mathbf{T}\mathbf{R}_{-45^\circ} &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}.
\end{aligned}$$

2. $\mathbf{T}\mathbf{p} = \mathbf{q} \Leftrightarrow \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_x \\ q_y \end{pmatrix} \Leftrightarrow \begin{pmatrix} t_{11}p_x + t_{12}p_y \\ t_{21}p_x + t_{22}p_y \end{pmatrix} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$. If $\mathbf{p}$ and $\mathbf{q}$ are to have the same magnitude, then

$$\begin{aligned}
&p_x^2 + p_y^2 = q_x^2 + q_y^2 \Leftrightarrow \\
&p_x^2 + p_y^2 = (t_{11}p_x + t_{12}p_y)^2 + (t_{21}p_x + t_{22}p_y)^2 \Leftrightarrow \\
&p_x^2 + p_y^2 = (t_{11}^2 + t_{21}^2)p_x^2 + (t_{12}^2 + t_{22}^2)p_y^2 + (t_{11}t_{12} + t_{21}t_{22})2p_x p_y \Leftrightarrow \\
&t_{11}^2 + t_{21}^2 = t_{12}^2 + t_{22}^2 = 1 \text{ and } t_{11}t_{12} + t_{21}t_{22} = 0.
\end{aligned}$$

These conditions imply that the column vectors in **T** are normalised and orthogonal. Hence **T** is an orthogonal matrix. Likewise, these are precisely the same conditions for the transpose of **T** to be its inverse, because

$$\mathbf{T}^t\mathbf{T} = \begin{pmatrix} t_{11} & t_{21} \\ t_{12} & t_{22} \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11}^2 + t_{21}^2 & t_{11}t_{12} + t_{21}t_{22} \\ t_{12}t_{11} + t_{22}t_{21} & t_{12}^2 + t_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \mathbf{T}^t = \mathbf{T}^{-1}.$$

3. Since $t_{11}t_{12} + t_{21}t_{22} = 0 \Leftrightarrow t_{22} = -t_{11}t_{12}/t_{21}, t_{21} \neq 0$, we find that $1 = t_{12}^2 + t_{22}^2 = t_{12}^2 + \left(t_{11}t_{12}/t_{21}\right)^2 = t_{12}^2/t_{21}^2\left(t_{21}^2 + t_{11}^2\right) = t_{12}^2/t_{21}^2$, that is $t_{21} = \pm t_{12}$ and $t_{22} = \mp t_{11}$.
But 2×2 rotation matrices do not allow for the upper sign option, so the conclusion is that orthogonal matrices represent a *broader* class than rotation matrices. The reason is that orthogonal matrices can include reflections as well as rotations.

4. $\mathbf{T}_1$ is orthogonal since $0.8^2 + 0.6^2 = 1$ and $0.8 \cdot 0.6 + (-0.6 \cdot 0.8) = 0.$ It is a rotation matrix with $\theta = -36.87^\circ$. $\mathbf{T}_2$ is orthogonal since $\left(\sqrt{3}/2\right)^2 + \left(1/2\right)^2 = 1$ and $-\sqrt{3}/2 \cdot 1/2 + 1/2 \cdot \sqrt{3}/2 = 0.$ However, $\mathbf{T}_2$ is not a a pure rotation matrix. $\mathbf{T}_3$ is not an orthogonal matrix. The column vectors are unit vectors but they are not orthogonal since $1/\sqrt{2} \cdot 1/\sqrt{2} + 1/\sqrt{2} \cdot 1/\sqrt{2} = 1 \neq 0$. Changing the sign on one of the entries in the matrix $\mathbf{T}_3$ would render it orthogonal.

5. $\mathbf{A}_1$ is orthogonal since all the column vectors are normalised and pair-wise orthogonal. $\mathbf{A}_2$ is not orthogonal. The column vectors are normalised but column vector 1 and 3 are not orthogonal. $\mathbf{A}_2$ would, however, be orthogonal if the sign of any one of the four fractional elements were reversed.

6. For an orthogonal matrix **O**, $\mathbf{OO}^t = \mathbf{I}$, where **I** is the identity matrix, so

   $$1 = \det \mathbf{I} = \det \mathbf{OO}^t = \det \mathbf{O} \cdot \det \mathbf{O}^t = (\det \mathbf{O})^2$$

   where the last step follows because $\det \mathbf{O}^t = \det \mathbf{O}$, see determinant property 7 on FS 6.

   The conclusion is that $\left(\det \mathbf{O}\right)^2 = 1 \Leftrightarrow \det \mathbf{O} = \pm 1$.