Problems for Lecture 15: Orthogonal Transformations

1. Show that an extension by a factor 2 along a line at 45° to the *x* and *y* axes (i.e., parallel to the direction vector $\mathbf{i}+\mathbf{j}$) is represented by the matrix transformation $\begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$.

2. The matrix $\mathbf{T}=\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ transforms the vector $\mathbf{p}=\begin{pmatrix} p_x \\ p_y \end{pmatrix}$ into the vector $\mathbf{q}=\begin{pmatrix} q_x \\ q_y \end{pmatrix}$.

   If the magnitude of the two vectors are the same $|\mathbf{p}|=|\mathbf{q}|$, what conditions between the elements $t_{11},t_{12},t_{21},t_{22}$ must be satisfied?

   Hence show that, under these conditions, the inverse of $\mathbf{T}$ is its transpose, that is, $\mathbf{T}^{-1}=\mathbf{T}^t$. [*HINT:* See Classwork 5, question (d).]

3. Matrices that satisfy the conditions of question 2 are called *orthogonal matrices* (see Classwork 5 for definition). Can all 2×2 orthogonal matrices be represented by rotation matrices of the form $\mathbf{R}_\theta=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$?

4. Determine whether the following 2×2 matrices are orthogonal:

   (a) $\mathbf{T}_1=\begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$, (b) $\mathbf{T}_2=\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$, (c) $\mathbf{T}_3=\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.

   Consider your conclusions in the light of questions 2 and 3.

5. One of the following 3×3 matrices is orthogonal, and the other could be made so by a single sign change. Which is which, and what sign change would be necessary?

   (a) $\mathbf{A}_1=\begin{pmatrix} 1/\sqrt{2} & 1/2 & -1/2 \\ -1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ (b) $\mathbf{A}_2=\begin{pmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & -4/5 \end{pmatrix}$.

6. If $\mathbf{O}$ is an orthogonal matrix, show that $\det\mathbf{O}=\pm1$.

   [*HINT*: Use the fact that $\det(\mathbf{AB})=(\det\mathbf{A})\cdot(\det\mathbf{B})$, where $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order. Remember that the inverse of an orthogonal matrix $\mathbf{O}$ is equal to its transpose, $\mathbf{O}^{-1}=\mathbf{O}^t$. One other property of determinants is also needed, see FS 6.]