Problems for Lecture 14: Answers

1. (a) A normal vector to the plane is given by the coefficients of the three unknowns, that is, $\mathbf{n}_1 = 5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$. To determine a unit normal vector, we need to divide by the magnitude of $\mathbf{n}_1$, $|\mathbf{n}_1| = \sqrt{5^2 + (-4)^2 + (-3)^2} = \sqrt{50}$, so the unit normal vector to the plane $\hat{\mathbf{n}}_1 = \frac{5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}}{\sqrt{50}} = \frac{5}{\sqrt{50}}\mathbf{i} - \frac{4}{\sqrt{50}}\mathbf{j} - \frac{3}{\sqrt{50}}\mathbf{k}$.

   (b) Dividing the equation for the plane by the magnitude of the normal vector yields $\frac{5x - 4y - 3z}{\sqrt{50}} = \frac{10}{\sqrt{50}} = \sqrt{2}$. In this form, the right-hand-side is the minimal distance from the origin to the plane, that is, $d_O = \sqrt{2}$, see Fact Sheet 4 or Fact Sheet 10.

   (c) Choose any point A on the plane, say $\overrightarrow{OA} = (2,0,0)$, found by inserting $y = z = 0$ into the equation for the plane and solving the resulting equation $5x = 10$. The vector from A to P is $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-1,3,5) = -\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$. Hence, the minimal distance from the point P to the plane $d_P = \left|\overrightarrow{AP} \bullet \hat{\mathbf{n}}\right| = \left|\frac{(-1)\cdot 5 + 3\cdot(-4) + 5\cdot(-3)}{\sqrt{50}}\right| = \frac{32}{\sqrt{50}} \approx 4.53$.

2. By inspection of the equation for the second plane, we see that a normal vector is $\mathbf{n}_2 = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. A direction vector $\mathbf{d}$ for the line of intersection of the two planes is therefore $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -4 & -3 \\ -2 & 1 & 1 \end{vmatrix} = -\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

   A point $\mathbf{r}_0$ that lies on both planes, and therefore on the line of intersection, can be found, for example, by setting $z = 0$ and solving the two resulting equations $5x - 4y = 10$ and $-2x + y = 2$ simultaneously, yielding $x = -6, y = -10$ so that $\mathbf{r}_0 = -6\mathbf{i} - 10\mathbf{j}$. The vector equation of the line of intersection is therefore given by $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d} = (-6\mathbf{i} - 10\mathbf{j}) + \lambda(-\mathbf{i} + \mathbf{j} - 3\mathbf{k})$. Solving the three associated component equiations w.r.t. λ, we find the Cartesian form $\lambda = \frac{x+6}{-1} = \frac{y+10}{1} = \frac{z}{-3}$.

3. The equation for the third plane $x - 2y - z = 14$ is a linear combination of the equations of the other two planes, namely the first plus twice the second. Therefore, in the sense of solving 3 equations with 3 unknowns, the third equation is redundant and the system of linear equations will have the same solutions as before, that is, the line of intersection determined in question 2.

4. The line joining the two points has direction $\mathbf{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (7,4,3) = 7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and it follows that a unit vector in the direction of the line is $\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|} = \frac{7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}}{\sqrt{74}}$. The vector from, say, A to P is $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. The minimal distance from the point P to the plane is

$$d = \left|\overrightarrow{AP} \times \hat{\mathbf{d}}\right| = \frac{1}{\sqrt{74}} \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & -2 \\ 7 & 4 & 3 \end{matrix} \right\| = \left|\frac{-\mathbf{i} - 23\mathbf{j} + 33\mathbf{k}}{\sqrt{74}}\right| = \sqrt{\frac{1619}{74}} \approx 4.68 .$$

5. We find a vector $\overrightarrow{A_1A_2}$ joining arbitrary points A_1 from line 1 and A_2. Using $\overrightarrow{OA_1} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\overrightarrow{OA_2} = \alpha\mathbf{i}+\mathbf{j}+\mathbf{k}$, we find $\overrightarrow{A_1A_2} = \overrightarrow{OA_2} - \overrightarrow{OA_1} = (\alpha-1)\mathbf{i}-\mathbf{j}-2\mathbf{k}$. The two lines will intersect when $\overrightarrow{A_1A_2}$ and the two direction vectors $\mathbf{d}_1 = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{d}_2 = 2\mathbf{i}-3\mathbf{j}-4\mathbf{k}$ are coplanar, that is, when

$$\overrightarrow{A_1A_2}\bullet(\mathbf{d}_1\times\mathbf{d}_2) = \det\left(\overrightarrow{A_1A_2},\mathbf{d}_1,\mathbf{d}_2\right) = \begin{vmatrix}\alpha-1 & 3 & 2\\ -1 & 2 & -3\\ -2 & 1 & -4\end{vmatrix} = \begin{vmatrix}\alpha+5 & 0 & 14\\ 3 & 0 & 5\\ -2 & 1 & -4\end{vmatrix} = -1\cdot\begin{vmatrix}\alpha+5 & 14\\ 3 & 5\end{vmatrix} = 0,$$

that is, when $5\alpha+25-42=0 \Leftrightarrow \alpha = 17/5$.

6. (a) In matrix form $\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}2 & 3\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ so the transformation is $\mathbf{T}_a = \begin{pmatrix}2 & 3\\ 1 & -1\end{pmatrix}$

(b) On matrix form $\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}7 & -4\\ 2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$, that is $x' = 7x-4y;\ \ y' = 2x.$

7. (a) $(2, \frac{1}{2})$. The associated transformation on matrix form $\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2}\end{pmatrix}$.

(b) (6, 3). The associated transformation on matrix form $\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$.

(c) $(2,-1)$. The associated transformation on matrix form $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

8. (a) $\mathbf{R}^z_\theta = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$ (b) $\mathbf{R}^x_\theta = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$ (c) $\mathbf{R}^y_\theta = \begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}$.

9. (a) $\mathbf{R}^z_{+45^\circ}\mathbf{r} = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{3}{\sqrt{2}}\\ 3\end{pmatrix}$ (b) $\mathbf{R}^z_{-45^\circ}\mathbf{r} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 3\end{pmatrix}$.

The magnitude in invariant since $\sqrt{(-\frac{1}{\sqrt{2}})^2+(\frac{3}{\sqrt{2}})^2+3^2} = \sqrt{(\frac{3}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2+3^2} = \sqrt{14}$.

10. $$\mathbf{R}^x_{-45^\circ}\mathbf{R}^y_{+45^\circ}\mathbf{r} = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} = \begin{pmatrix}2\sqrt{2}\\ 1+\sqrt{2}\\ 1-\sqrt{2}\end{pmatrix}.$$