Problems for Lectures 14: Lines, Planes, Rotations

1. A plane in $\mathbb{R}^3$ is defined by the equation $5x - 4y - 3z = 10$. Find
   (a) the unit normal vector $\hat{\mathbf{n}}_1$,
   (b) the minimal (shortest, perpendicular) distance, d_O, from the origin to the plane,
   (c) the mininam distance, d_P, from the point $\overrightarrow{OP} = (1,3,5)$ to the plane.

2. Consider the two planes $5x - 4y - 3z = 10$ and $-2x + y + z = 2$. Find a normal vector $\mathbf{n}_2$ to the second plane (you found a normal to the first in question 1). Hence, find an equation for the *line of intersection* of the two planes in both vector and Cartesian form.

3. What can you say about the intersection of a third plane defined by $x - 2y - z = 14$ with the two planes specified in question 2?

4. Find the minimal distance from the point $\overrightarrow{OP} = (1,-2,0)$ to the line joining the two points $\overrightarrow{OA} = (-2,1,2)$ and $\overrightarrow{OB} = (5,5,5)$.

5. For which value of α do the two lines given by $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(3\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \lambda \in \mathbb{R}$ and $\mathbf{r} = (\alpha\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}), \mu \in \mathbb{R}$ intersect?

6. Consider $\mathbb{R}^2$. (a) Write down the matrix of the transformation defined by $\begin{matrix} x' = 2x + 3y \\ y' = x - y \end{matrix}$.
   (b) Write down the equations for the transformation whose matrix is $\mathbf{T} = \begin{pmatrix} 7 & -4 \\ 2 & 0 \end{pmatrix}$.

7. State the transformed position of the point (2, 1) under the following transformations:
   (a) Contraction (shrinkage) of factor 2 in the *y*-direction,
   (b) Extension (enlargement) of factor 3 in the x- and y-direction.
   (c) Reflection in the *x*-axis.

8. The 3×3 matrix for a rotation of angle θ about the z-axis is $\mathbf{R}_\theta^z = \begin{pmatrix} \cos\theta & \mp\sin\theta & 0 \\ \pm\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (a) Which signs apply if a positive θ corresponds to anti-clockwise rotation about the positive *z*-axis in a right-handed coordinate system?

   Find the analogous matrices for

   (b) an anti-clockwise rotation about the positive *x*-axis, $\mathbf{R}_\theta^x$,

   (c) an anti-clockwise rotation about the positive *y*-axis, $\mathbf{R}_\theta^y$.

9. Find the resulting vector if $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is rotated about the +*z*-axis by (a) 45° anti-clockwise and (b) 45° clockwise. Check that the magnitude remains invariant.

10. Using the same sign-convention as in question 8, the vector in the previous question is rotated first by 45° anti-clockwise about the +*y*-axis and then by 45° clockwise about the +*x*-axis. Find the new vector. Check, once again, that the operation preserves the magnitude of the vector.