Problems for Lecture 13: Answers

Consider the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{A}$ is an $n \times n$ matrix of coefficients.

The homogeneous equation always has the trivial solution, $\mathbf{x} = \mathbf{0}$.

If $\det \mathbf{A} \neq 0$, the homogeneous equation has *only the trivial solution*, $\mathbf{x} = \mathbf{0}$.

If $\det \mathbf{A} = 0$, the homogeneous equation has *additional non-trivial solutions*, $\mathbf{x} \neq \mathbf{0}$.

1. (a) The determinant of the matrix of coefficients $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 12 - 10 = 2 \neq 0$. Therefore,

$x = y = 0$ is the only solution to the homogeneous eq. It has no non-trivial solutions.

(b) The determinant of the matrix of coefficients $\begin{vmatrix} 3 & -5 \\ 7 & 2 \end{vmatrix} = 6 + 35 = 41 \neq 0$. Therefore,

the homogeneous equation has no non-trivial solutions. The only solution is $x = y = 0$.

(c) The determinant of the matrix of coefficients $\begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 12 - 12 = 0$. The two

equations are proportional: Eq.(1) is $1.5\times$ Eq.(2). Hence, one of the equations is redundant, say Eq.(2). Multiplying Eq.(1) by $\frac{1}{3}$ yields $2x + y = 0$, which defines the line of solutions (non-trivial as well as the trivial solution). On parametric form, the equation for the line of solutions is $\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$.

(d) The determinant of the matrix of coefficients $\begin{vmatrix} 1.4 & -1.2 \\ -2.1 & 1.8 \end{vmatrix} = 2.52 - 2.52 = 0$. The

two equations are proportional: Eq.(2) is $-1.5\times$ Eq.(1) and therefore one of the equations is redundant, say Eq.(1). Multiplying Eq.(2) by $\frac{1}{1.8}$ yields $-\frac{7}{6}x + y = 0$ which defines the line of solutions. On parametric form, the equation for the line of solutions is $\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \frac{7}{6} \end{pmatrix}, \lambda \in \mathbb{R}$.

2. (a) The determinant of the matrix of coefficients $\begin{vmatrix} 8 & 1 & 8 \\ 6 & 4 & 4 \\ 5 & -1 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 6 & 4 & 4 \\ 5 & -1 & 6 \end{vmatrix} = 0$ where

we have added $(-1)\times$row 3 to row 1 and $-\frac{1}{2}\times$row 2 to row 1. Hence, the system has non-trivial solutions in addition to the trivial solution.

(b) The determinant of the matrix of coefficients is easily found by adding $2\times$row 2 to row 1 and expanding by column 2: $\begin{vmatrix} 5 & 2 & 2 \\ 1 & -1 & 4 \\ 7 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 0 & 10 \\ 1 & -1 & 4 \\ 7 & 0 & 1 \end{vmatrix} = -1\cdot\begin{vmatrix} 7 & 10 \\ 7 & 1 \end{vmatrix} = 63 \neq 0$.

Hence, the homogeneous equation has only the trivial solution $p = q = r = 0$. No non-trivial solutions exist.

(c) The determinant of the matrix of coefficients $\begin{vmatrix} 12 & -16 & 2 & 8 \\ -6 & 6 & 14 & -3 \\ 10 & 10 & -7 & -5 \\ 11 & -18 & 2 & 9 \end{vmatrix} = 0$ since column 2 and column 4 are proportional (column 2 = $-2\times$column 4). Hence the homogeneous equation has non-trivial solutions in addition to the trivial solution $\mathbf{x} = \mathbf{0}$.

3. Let us first evaluate the vector products:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 7 & -2 & 4 \end{vmatrix} = (4-6)\mathbf{i} - (8+21)\mathbf{j} + (-4-7)\mathbf{k} = -2\mathbf{i} - 29\mathbf{j} - 11\mathbf{k},$$

$$\mathbf{C}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 5 \\ 7 & -2 & 4 \end{vmatrix} = (0+10)\mathbf{i} - (16-35)\mathbf{j} + (-8-0)\mathbf{k} = 10\mathbf{i} + 19\mathbf{j} - 8\mathbf{k},$$

$$\mathbf{C}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 5 \\ 2 & 1 & -3 \end{vmatrix} = (0-5)\mathbf{i} - (-12-10)\mathbf{j} + (4-0)\mathbf{k} = -5\mathbf{i} + 22\mathbf{j} + 4\mathbf{k},$$

$$\mathbf{B}\times\mathbf{C} = -\mathbf{C}\times\mathbf{B} = -10\mathbf{i} - 19\mathbf{j} + 8\mathbf{k}.$$

Hence, we find that

(a) $(\mathbf{A}\times\mathbf{B})\bullet\mathbf{C} = \left(-2\mathbf{i} - 29\mathbf{j} - 11\mathbf{k}\right)\bullet\left(4\mathbf{i} + 5\mathbf{k}\right) = (-2)\cdot 4 + (-29)\cdot 0 + (-11)\cdot 5 = -63.$

(b) $\mathbf{A}\bullet\left(\mathbf{C}\times\mathbf{B}\right) = \left(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}\right)\bullet\left(10\mathbf{i} + 19\mathbf{j} - 8\mathbf{k}\right) = 2\cdot 10 + 1\cdot 19 + (-3)\cdot(-8) = 63.$

(c) $\mathbf{B}\bullet\left(\mathbf{C}\times\mathbf{A}\right) = \left(7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}\right)\bullet\left(-5\mathbf{i} + 22\mathbf{j} + 4\mathbf{k}\right) = 7\cdot(-5) + (-2)\cdot 22 + 4\cdot 4 = -63.$

Note that the magnitudes are all the same, but the sign changes when cyclic order is not maintained. See Fact Sheet 9.

(d) $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -29 & -11 \\ 4 & 0 & 5 \end{vmatrix} = -145\mathbf{i} - (-10+44)\mathbf{j} + 116\mathbf{k} = -145\mathbf{i} - 34\mathbf{j} + 116\mathbf{k}$,

(e) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ -10 & -19 & 8 \end{vmatrix} = -49\mathbf{i} + 14\mathbf{j} - 28\mathbf{k}$.

4. (a) Let us consider the equation $c_1\mathbf{A} + c_2\mathbf{B} + c_3\mathbf{C} = \mathbf{0}$. On component form we have

$$\begin{aligned} 2c_1 + 7c_2 + 4c_3 &= 0 \\ c_1 - 2c_2 &= 0 \\ -3c_1 + 4c_2 + 5c_3 &= 0 \end{aligned}$$

. However, the determinant of the matrix of coefficient, whose columns are $\mathbf{A}, \mathbf{B},$ and $\mathbf{C}$, respectively, $\det(\mathbf{A},\mathbf{B},\mathbf{C}) = \mathbf{A}\bullet(\mathbf{B}\times\mathbf{C}) = -63 \neq 0$ so the trivial solution $c_1 = c_2 = c_3 = 0$ is the only solution. Hence, the vectors $\mathbf{A},\mathbf{B},\mathbf{C}$ are linearly independent.

(b) *Linear dependence* implies that the three vectors are in the same plane; they are coplanar. *Linear independence* implies that the three vectors are <u>not</u> coplanar. The determinant formed from the components of the vectors $\mathbf{A},\mathbf{B},\mathbf{C}$ is the triple scalar product, that is, $\det(\mathbf{A},\mathbf{B},\mathbf{C}) = \mathbf{A}\bullet(\mathbf{B}\times\mathbf{C})$ and represents the volume of the parallelepiped whose sides are given by the three vectors. The volume is zero when the vectors are coplanar, that is, when the vectors are linearly dependent. The volume is non-zero when the vectors are linearly independent.

(c) The determinant with columns $\mathbf{A},\mathbf{B},\mathbf{C}$, expanded by the second row is

$$\det(\mathbf{A},\mathbf{B},\mathbf{C}) = \begin{vmatrix} 2+\alpha & 7 & 4 \\ 1 & -2 & 0 \\ -3 & 4 & 5 \end{vmatrix} = -1\cdot\begin{vmatrix} 7 & 4 \\ 4 & 5 \end{vmatrix} - 2\cdot\begin{vmatrix} 2+\alpha & 4 \\ -3 & 5 \end{vmatrix} = -63 - 10\alpha$$

. Therefore,

$\det(\mathbf{A},\mathbf{B},\mathbf{C}) = 0 \Leftrightarrow \alpha = 6.3$, making the vectors linearly dependent.

5. From Fact Sheet 9, we have the **B**AC-**C**AB rule $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\bullet\mathbf{C}) - \mathbf{C}(\mathbf{A}\bullet\mathbf{B})$.

Hence, by applying this rule on the three triple vector products, we find

$$\begin{aligned} &\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = \\ &\mathbf{B}(\mathbf{A}\bullet\mathbf{C}) - \mathbf{C}(\mathbf{A}\bullet\mathbf{B}) + \mathbf{C}(\mathbf{B}\bullet\mathbf{A}) - \mathbf{A}(\mathbf{B}\bullet\mathbf{C}) + \mathbf{A}(\mathbf{C}\bullet\mathbf{B}) - \mathbf{B}(\mathbf{C}\bullet\mathbf{A}) = 0. \end{aligned}$$