Problems for Lecture 13

Homogeneous Equations, Triple Products, Linear Independence

1. Determine which of following pairs of homogeneous equations have a non-trivial solution and, for those that do, find the equation of the line that represents the solution.

(a) $\begin{matrix} 3x+5y=0 \\ 2x+4y=0, \end{matrix}$ (b) $\begin{matrix} 3x-5y=0 \\ 7x+2y=0, \end{matrix}$ (c) $\begin{matrix} 6x+3y=0 \\ 4x+2y=0, \end{matrix}$ (d) $\begin{matrix} 1.4x-1.2y=0 \\ -2.1x+1.8y=0. \end{matrix}$

2. Which of following sets of homogeneous equations have a non-trivial solution?

(a) $\begin{matrix} 8x+y+8z=0 \\ 6x+4y+4z=0 \\ 5x-y+6z=0, \end{matrix}$ (b) $\begin{matrix} 5p+2q+2r=0 \\ p-q+4r=0 \\ 7p+r=0, \end{matrix}$ (c) $\begin{matrix} 12x_1-16x_2+2x_3+8x_4=0 \\ -6x_1+6x_2+14x_3-3x_4=0 \\ 10x_1+10x_2-7x_3-5x_4=0 \\ 11x_1-18x_2+2x_3+9x_4=0. \end{matrix}$

3. Let $\mathbf{A} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$, $\mathbf{B} = 7\mathbf{i}-2\mathbf{j}+4\mathbf{k}$, and $\mathbf{C} = 4\mathbf{i}+5\mathbf{k}$.

Find the triple scalar products (a) $(\mathbf{A}\times\mathbf{B})\bullet\mathbf{C}$, (b) $\mathbf{A}\bullet(\mathbf{C}\times\mathbf{B})$, (c) $\mathbf{B}\bullet(\mathbf{C}\times\mathbf{A})$.

Find the triple vector products (d) $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$, (e) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

4. Three vectors **A**, **B**, and **C** are *linearly dependent* if it is possible to write one of the vectors as a linear combination of the other two, for example, $\mathbf{A} = p\mathbf{B}+q\mathbf{C}$.

Three vectors are *linearly independent* if it is <u>not</u> possible to do this.

More generally, we say that a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is *linearly dependent* if there exists numbers $c_1, c_2, \ldots, c_n$ **not all** equal to zero such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$ (which, of course implies that one of the vectors with a non-zero coefficient can be written as a linear combination of (a subset of) the other vectors.

The set of vectors is *linearly independent* if $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$ implies all coefficients are zero, $c_1 = c_2 = \cdots c_n$.

(a) Show that the vectors $\mathbf{A},\mathbf{B},\mathbf{C}$ defined in question 3 are linearly independent.

(b) What does linear dependence or independence imply about the determinant formed from the components of the three vectors?

(c) Redefine $\mathbf{A} = (2+\alpha)\mathbf{i}+\mathbf{j}-3\mathbf{k}$. What value of α makes $\mathbf{A},\mathbf{B},\mathbf{C}$ linearly dependent?

5. Show the identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=0$ for any three vectors **A**, **B**, and **C.** [*HINT:* Use the formula at the bottom of Fact Sheet 9.]