Problems for Lecture 12: Answers

1. (a) Since $\det \mathbf{A}_1 = \begin{vmatrix} 0 & 2 \\ -2 & 4 \end{vmatrix} = 4 \neq 0$, the matrix $\mathbf{A}_1$ is non-singular. We find the inverse

$$\mathbf{A}_1^{-1} = \frac{\text{adj } \mathbf{A}_1}{\det \mathbf{A}_1} = \frac{1}{\det \mathbf{A}_1}\begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}^t = \frac{1}{4}\begin{pmatrix} 4 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Please check that $\mathbf{A}_1^{-1}\mathbf{A}_1 = \mathbf{A}_1\mathbf{A}_1^{-1} = \mathbf{I}$ ☺.

(b) Since $\det \mathbf{A}_2 = \begin{vmatrix} 6 & -4 \\ -3 & 2 \end{vmatrix} = 12 - (-3)\cdot(-4) = 0$, the matrix $\mathbf{A}_2$ is singular.

(c) Since $\det \mathbf{A}_3 = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1 \neq 0$, the matrix $\mathbf{A}_3$ is non-singular. We find the inverse

$$\mathbf{A}_3 = \frac{\text{adj } \mathbf{A}_3}{\det \mathbf{A}_3} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}^t = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}.$$ Again check that $\mathbf{A}_3^{-1}\mathbf{A}_3 = \mathbf{A}_3\mathbf{A}_3^{-1} = \mathbf{I}$ ☺.

2 (a) The matrix $\mathbf{B}_1$ is singular. Column 2 is $2\times$ column 3, implying $\det \mathbf{B}_1 = 0$.

(b) Expanding by the first row we find $\det \mathbf{B}_2 = -7 \cdot \begin{vmatrix} 3 & 6 \\ 5 & 2 \end{vmatrix} = -7\cdot(-24) = 168 \neq 0$ so the matrix $\mathbf{B}_2$ is non-singular.

(c) The matrix $\mathbf{B}_3$ is singular. Row 3 is ($3\times$ row 1 + $2\times$ row 2), so $\det \mathbf{B}_3 = 0$.

(d) Adding $(-1)\times$ column 2 to column 1 and expanding by column 1 we find

$$\det \mathbf{B}_4 = \begin{vmatrix} 4 & 4 & 4 \\ 2 & 1 & 2 \\ -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 4 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = -1 \cdot \begin{vmatrix} 4 & 4 \\ -1 & 1 \end{vmatrix} = -8 \neq 0,$$ so $\mathbf{B}_4$ is non-singular.

(e) The matrix $\mathbf{B}_5$ is singular. Rows 1 and 3 are identical, implying $\det \mathbf{B}_5 = 0$.

(f) The matrix $\mathbf{B}_6$ is not square, so it has *no* determinant. Therefore, in principle, the issue of singularity does not arise. However, the term *singular* is sometimes applied to <u>*any*</u> matrix that has no inverse. Hence, non-square matrices are singular, since they, by definition, have no inverse.

3. The matrix of the coefficients is $\mathbf{A} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix}$.

(a) Adding $2\times$ row 1 to row 2, $(-1)\times$ row 1 to row 3, and expanding by column 1:

$$\det \mathbf{A} = \begin{vmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 5 & 2 \\ 0 & -4 & -2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 5 & 2 \\ -4 & -2 \end{vmatrix} = -1\cdot(-10+8) = 2 \neq 0.$$

(b) The ij th element in the matrix of cofactors is $C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}$. Hence, we find

$$\mathbf{C} = \begin{pmatrix} \begin{vmatrix} 1 & -4 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ -1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -7 & 2 & -3 \\ -8 & 2 & -4 \\ -11 & 2 & -5 \end{pmatrix}$$

(c) The adjoint matrix is the transpose of the matrix of cofactors. Hence, we have

$$\operatorname{adj} \mathbf{A} = \mathbf{C}^t = \begin{pmatrix} -7 & 2 & -3 \\ -8 & 2 & -4 \\ -11 & 2 & -5 \end{pmatrix}^t = \begin{pmatrix} -7 & -8 & -11 \\ 2 & 2 & 2 \\ -3 & -4 & -5 \end{pmatrix}.$$

(d) The inverse $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}\mathbf{A}}{\det \mathbf{A}} = \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix}$.

(e) Since $\det \mathbf{A} \neq 0$, $\mathbf{A}$ is invertible. Therefore, the general solution to the system of linear equation can be found using $\mathbf{Ax} = \mathbf{k} \Leftrightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{k} \Leftrightarrow \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{k} \Leftrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{k}$

Hence $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -\frac{7}{2}k_1 - 4k_2 - \frac{11}{2}k_3 \\ k_1 + k_2 + k_3 \\ -\frac{3}{2}k_1 - 2k_2 - \frac{5}{2}k_3 \end{pmatrix}$.

Finally, we check (belatedly) that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$:

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} - 8 + \frac{11}{2} & -7-4+11 & -\frac{21}{2} + 16 - \frac{11}{2} \\ -1+2-1 & 2+1-2 & 3-4+1 \\ \frac{3}{2} - 4 + \frac{5}{2} & -3-2+5 & -\frac{9}{2} + 8 - \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{2} + 2 - \frac{9}{2} & 4+2-6 & \frac{11}{2} + 2 - \frac{15}{2} \\ -7+1+6 & -8+1+8 & -11+1+10 \\ \frac{7}{2} - 2 - \frac{3}{2} & 4-2-2 & \frac{11}{2} - 2 - \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. (a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} - 4 + \frac{11}{2} \\ 1+1-1 \\ -\frac{3}{2} - 2 + \frac{5}{2} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{35}{2} + 32 \\ 5-8 \\ -\frac{15}{2} + 16 \end{pmatrix} = \begin{pmatrix} \frac{29}{2} \\ -3 \\ \frac{17}{2} \end{pmatrix}$ (c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{21}{2} - 16 - \frac{55}{2} \\ 3+4+5 \\ -\frac{9}{2} - 8 - \frac{25}{2} \end{pmatrix} = \begin{pmatrix} -54 \\ 12 \\ -25 \end{pmatrix}$.

5. Adding $(-5) \times$ row 1 to row 2, adding row 3 to row 1, and expanding by row 2 we find:

$$\det \mathbf{D} = \begin{vmatrix} 0 & 4 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 3 & 2 & -1 \\ 3 & -2 & 5 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 0 & 3 & 0 \\ -1 & 2 & -1 \\ 3 & 5 & 4 \end{vmatrix} = 2 \cdot (-3) \cdot \begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix} = 6.$$