Problems for Lecture 12
Singular Matrices and Linear Equations

1. Which of the following matrices are non-singular? For each that is, find the inverse:

(a) $\mathbf{A}_1 = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$, (b) $\mathbf{A}_2 = \begin{pmatrix} 6 & -4 \\ -3 & 2 \end{pmatrix}$, (c) $\mathbf{A}_3 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$.

2. Which of the following matrices are singular? In those cases where a matrix can be seen *by inspection* to be singular, give your reasoning.

(a) $\mathbf{B}_1 = \begin{pmatrix} 8 & 6 & 3 \\ 5 & 8 & 4 \\ 5 & 4 & 2 \end{pmatrix}$, (b) $\mathbf{B}_2 = \begin{pmatrix} 0 & 7 & 0 \\ 3 & -5 & 6 \\ 5 & 4 & 2 \end{pmatrix}$, (c) $\mathbf{B}_3 = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 4 & -1 \\ -1 & 14 & 1 \end{pmatrix}$,

(d) $\mathbf{B}_4 = \begin{pmatrix} 4 & 4 & 4 \\ 2 & 1 & 2 \\ -1 & -1 & 1 \end{pmatrix}$, (e) $\mathbf{B}_5 = \begin{pmatrix} 3.5 & -7.2 & 2.1 & 4.4 \\ 5.3 & 6.2 & 0 & -6.2 \\ 3.5 & -7.2 & 2.1 & 4.4 \\ 1.7 & 0 & -5.3 & 0 \end{pmatrix}$, (f) $\mathbf{B}_6 = \begin{pmatrix} 3 & -5 & 0 & -1 \\ 2 & 1 & 7 & 4 \\ 0 & 6 & -4 & 2 \end{pmatrix}$.

3. Consider the system of linear equations of 3 equations with 3 unknowns x_1, x_2, x_3:

$$\begin{aligned} -x_1 + 2x_2 + 3x_3 &= k_1 \\ 2x_1 + x_2 - 4x_3 &= k_2 \\ -x_1 - 2x_2 + x_3 &= k_3 \end{aligned} \quad \Leftrightarrow \quad \mathbf{Ax} = \mathbf{k}$$

Write down the matrix of the coefficients $\mathbf{A}$ and find in sequence

(a) the determinant $\det \mathbf{A}$,

(b) the matrix of the cofactors $\mathbf{C}$,

(c) the adjoint matrix $\operatorname{adj}\mathbf{A}$,

(d) the inverse $\mathbf{A}^{-1}$,

(e) the general solution to the system of linear equations.

Check that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the 3×3 identity matrix.

4. Obtain the solution of the equations in question 3 for the following values of $\mathbf{k}$:

(a) $\mathbf{k} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, (b) $\mathbf{k} = \begin{pmatrix} 5 \\ -8 \\ 0 \end{pmatrix}$, (c) $\mathbf{k} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$.

5. Find the determinant of the matrix $\mathbf{D} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 7 & 5 & 5 \\ -1 & 3 & 2 & -1 \\ 3 & -2 & 5 & 4 \end{pmatrix}$.