Problems to Lecture 11: Answers

1. We use the definition of a matrix products given on Fact Sheet 7 and notice that when $\mathbf{A}$ is an $m\times p$ matrix and $\mathbf{B}$ is an $p\times n$ matrix, then $\mathbf{AB}$ is an $m\times n$ matrix.

(a) $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}=1\cdot5+2\cdot6=17$, (b) $\begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}=3\cdot5+4\cdot6=39$,

(c) $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}=\begin{pmatrix}1\cdot5+2\cdot6\\3\cdot5+4\cdot6\end{pmatrix}=\begin{pmatrix}17\\39\end{pmatrix}$, (d) $\begin{pmatrix}5\\6\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix}=\begin{pmatrix}5\cdot1 & 5\cdot2\\6\cdot1 & 6\cdot2\end{pmatrix}=\begin{pmatrix}5 & 10\\6 & 12\end{pmatrix}$,

(e) $\begin{pmatrix}1 & -3 & 5 & -7\end{pmatrix}\begin{pmatrix}8\\6\\4\\2\end{pmatrix}=1\cdot8-3\cdot6+5\cdot4-7\cdot2=-4$,

(f) $\begin{pmatrix}8\\6\\4\\2\end{pmatrix}\begin{pmatrix}1 & -3 & 5 & -7\end{pmatrix}=\begin{pmatrix}8\cdot1 & 8\cdot(-3) & 8\cdot5 & 8\cdot(-7)\\6\cdot1 & 6\cdot(-3) & 6\cdot5 & 6\cdot(-7)\\4\cdot1 & 4\cdot(-3) & 4\cdot5 & 4\cdot(-7)\\2\cdot1 & 2\cdot(-3) & 2\cdot5 & 2\cdot(-7)\end{pmatrix}=\begin{pmatrix}8 & -24 & 40 & -56\\6 & -18 & 30 & -42\\4 & -12 & 20 & -28\\2 & -6 & 10 & -14\end{pmatrix}$.

2. The 2×2 matrix that represent a rotation in $\mathbb{R}^2$ about the origin by some angle θ, is given by $\mathbf{R}_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$, where we define the *positive* rotation direction of rotation as *anti-clockwise* (and hence the negative direction of rotation as clockwise).

(a)

$$\mathbf{R}_{+45^\circ}=\begin{pmatrix}\cos45^\circ & -\sin45^\circ\\\sin45^\circ & \cos45^\circ\end{pmatrix}=\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\-1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix};$$

$$\mathbf{R}_{+90^\circ}=\begin{pmatrix}\cos90^\circ & -\sin90^\circ\\\sin90^\circ & \cos90^\circ\end{pmatrix}=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}.$$

(b) $\mathbf{R}^2_{+45^\circ}=\mathbf{R}_{+45^\circ}\mathbf{R}_{+45^\circ}=\frac{1}{2}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0 & -2\\2 & 0\end{pmatrix}=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}=\mathbf{R}_{+90^\circ}$.

(*c*) $\mathbf{R}^2_{+90^\circ}=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}=\mathbf{R}_{+180^\circ}$; $\mathbf{R}^2_{-90^\circ}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}=\mathbf{R}_{-180^\circ}$.

$\mathbf{R}_{+180^\circ}=\mathbf{R}_{-180^\circ}$ correspond to axes inversion: both the *x*- and *y*-axis change direction.

3. (a) Since the rotation is clockwise, the angle is negative. Hence, we find

$$\mathbf{R}_{-|\theta|} = \begin{pmatrix} \cos\left(-\sin^{-1}(4/5)\right) & -\sin\left(-\sin^{-1}(4/5)\right) \\ \sin\left(-\sin^{-1}(4/5)\right) & \cos\left(-\sin^{-1}(4/5)\right) \end{pmatrix} = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

and we notice that the numerical value of $\theta = \sin^{-1}(4/5) = 0.927\ \boldsymbol{rad} = 53.13^\circ$.

(b) We find the matrix associated with 45° anti-clockwise rotation: $\mathbf{R}_{+45^\circ} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Therefore, the matrix representing a clockwise rotation of $\theta = \sin^{-1}(4/5)$ followed by an anti-clockwise rotation of 45° is

$$\mathbf{R_{net}} = \mathbf{R}_{+45^\circ}\mathbf{R}_{-|\theta|} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & 1 \\ -1 & 7 \end{pmatrix}$$

This represents a net *clockwise* rotation of 8.13°, that is, $\mathbf{R}_{-8.13^\circ}$.

(c) We find $\mathbf{R}_{-|\theta|}\mathbf{R}_{+45^\circ} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & 1 \\ -1 & 7 \end{pmatrix}$, which is the same as the result in 3(b). In general, matrix products of rotation matrices are commutative due to the nature of the operation.

(d) The inverse of the matrix $\mathbf{R}_{\text{net}} = \mathbf{R}_{-8.13^\circ}$ must be the reverse rotation (indeed, $\mathbf{R}_\theta^{-1} = \mathbf{R}_{-\theta}$), that is, $\mathbf{R}^{-1}_{\mathbf{net}} = \mathbf{R}_{+8.13^\circ} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & -1 \\ 1 & 7 \end{pmatrix}$. To confirm this, note that

$$\mathbf{R_{net}}\mathbf{R}^{-1}_{\mathbf{net}} = \frac{1}{50}\begin{pmatrix} 7 & 1 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 7 & -1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \&\ \mathbf{R}^{-1}_{\boldsymbol{net}}\mathbf{R}_{\boldsymbol{net}} = \frac{1}{50}\begin{pmatrix} 7 & -1 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. (a) All elements above the leading diagonal are zero. Therefore, the determinant is the product of the diagonal elements: $\det\mathbf{A} = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 \\ 5 & 4 & 3 & 0 \\ 3 & 2 & 1 & 2 \end{vmatrix} = 4\cdot(-1)\cdot 3\cdot 2 = -24$.

(b) We expand the determinant of the matrix $\mathbf{B}$ by the 4$^{\text{th}}$ column:

$$\det\mathbf{B} = 2\begin{vmatrix} 4 & 0 & 1 \\ 6 & -1 & 0 \\ 5 & 4 & 3 \end{vmatrix} = 2\begin{vmatrix} 4 & 0 & 1 \\ 6 & -1 & 0 \\ -7 & 4 & 0 \end{vmatrix} = 2\cdot 1\begin{vmatrix} 6 & -1 \\ -7 & 4 \end{vmatrix} = 2\cdot(24-7) = 34.$$

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Expanded by 4th column. Adding $(-3)\times$ ***row1 to row3. Expanded by 3rd column.***