Problems for Lecture 10: Answers

1.

(*a*) $3\mathbf{A} = 3\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ 12 & 18 \end{pmatrix}$,

(*b*) $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 9 & 10 \end{pmatrix}$,

(*c*) $3\mathbf{B} - 2\mathbf{C} = 3\begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix} - 2\begin{pmatrix} -4 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 15 & 12 \end{pmatrix} + \begin{pmatrix} 8 & -4 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 14 & -7 \\ 15 & 10 \end{pmatrix}$.

2.

(a) $\mathbf{r}_1 = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 3 + 3\cdot 4 \\ 4\cdot 3 + 6\cdot 4 \end{pmatrix} = \begin{pmatrix} 18 \\ 36 \end{pmatrix}$,

(b) $\mathbf{r}_2 = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 3 + (-1)\cdot 4 \\ 5\cdot 3 + 4\cdot 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 31 \end{pmatrix}$,

(c) $\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 = \begin{pmatrix} 18 \\ 36 \end{pmatrix} + \begin{pmatrix} 2 \\ 31 \end{pmatrix} = \begin{pmatrix} 20 \\ 67 \end{pmatrix}$,

(d) $\mathbf{r}_4 = \begin{pmatrix} 4 & 2 \\ 9 & 10 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4\cdot 3 + 2\cdot 4 \\ 9\cdot 3 + 10\cdot 4 \end{pmatrix} = \begin{pmatrix} 20 \\ 67 \end{pmatrix}$.

The transformation effected by the sum of the two matrices $(\mathbf{A} + \mathbf{B})$ is the same as the sum of the two transformations $\mathbf{A}$ and $\mathbf{B}$.

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Then $\mathbf{Ar} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$ and

$\mathbf{Br} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_{11}x + b_{12}y \\ b_{21}x + b_{22}y \end{pmatrix}$ so that

$\mathbf{Ar} + \mathbf{Br} = \begin{pmatrix} a_{11}x + a_{12}y + b_{11}x + b_{12}y \\ a_{21}x + a_{22}y + b_{21}x + b_{22}y \end{pmatrix}$, using the property of matrix addition

$= \begin{pmatrix} (a_{11} + b_{11})x + (a_{12} + b_{12})y \\ (a_{21} + b_{21})x + (a_{22} + b_{22})y \end{pmatrix}$, using the property for real numbers

$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, using then definition of matrix multiplication

$= \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix}$, using the definition of matrix addition

$= (\mathbf{A} + \mathbf{B})\mathbf{r}$.

3. (a) $$\mathbf{AB}=\begin{pmatrix}2&3\\4&6\end{pmatrix}\begin{pmatrix}2&-1\\5&4\end{pmatrix}=\begin{pmatrix}2\cdot2+3\cdot5&2\cdot(-1)+3\cdot4\\4\cdot2+6\cdot5&4\cdot(-1)+6\cdot4\end{pmatrix}=\begin{pmatrix}19&10\\38&20\end{pmatrix},$$

(b) $$\mathbf{BC}=\begin{pmatrix}2&-1\\5&4\end{pmatrix}\begin{pmatrix}-4&2\\0&1\end{pmatrix}=\begin{pmatrix}2\cdot(-4)+(-1)\cdot0&2\cdot2+(-1)\cdot1\\5\cdot(-4)+4\cdot0&5\cdot2+4\cdot1\end{pmatrix}=\begin{pmatrix}-8&3\\-20&14\end{pmatrix},$$

(c) $$\mathbf{CB}=\begin{pmatrix}-4&2\\0&1\end{pmatrix}\begin{pmatrix}2&-1\\5&4\end{pmatrix}=\begin{pmatrix}(-4)\cdot2+2\cdot5&(-4)\cdot(-1)+2\cdot4\\0\cdot2+1\cdot5&0\cdot(-1)+1\cdot4\end{pmatrix}=\begin{pmatrix}2&12\\5&4\end{pmatrix},$$

(d) $$\mathbf{AC}=\begin{pmatrix}2&3\\4&6\end{pmatrix}\begin{pmatrix}-4&2\\0&1\end{pmatrix}=\begin{pmatrix}2\cdot(-4)+3\cdot0&2\cdot2+3\cdot1\\4\cdot(-4)+6\cdot0&4\cdot2+6\cdot1\end{pmatrix}=\begin{pmatrix}-8&7\\-16&14\end{pmatrix},$$

(e) $$(\mathbf{AB})\mathbf{C}=\begin{pmatrix}19&10\\38&20\end{pmatrix}\begin{pmatrix}-4&2\\0&1\end{pmatrix}=\begin{pmatrix}19\cdot(-4)+10\cdot0&19\cdot2+10\cdot1\\38\cdot(-4)+20\cdot0&38\cdot2+20\cdot1\end{pmatrix}=\begin{pmatrix}-76&48\\-152&96\end{pmatrix},$$

(f) $$\mathbf{A}(\mathbf{BC})=\begin{pmatrix}2&3\\4&6\end{pmatrix}\begin{pmatrix}-8&3\\-20&14\end{pmatrix}=\begin{pmatrix}2\cdot(-8)+3\cdot(-20)&2\cdot3+3\cdot14\\4\cdot(-8)+6\cdot(-20)&4\cdot3+6\cdot14\end{pmatrix}=\begin{pmatrix}-76&48\\-152&96\end{pmatrix},$$

(g) $$(\mathbf{A}+\mathbf{B})\mathbf{C}=\begin{pmatrix}4&2\\9&10\end{pmatrix}\begin{pmatrix}-4&2\\0&1\end{pmatrix}=\begin{pmatrix}4\cdot(-4)+2\cdot0&4\cdot2+2\cdot1\\9\cdot(-4)+10\cdot0&9\cdot2+10\cdot1\end{pmatrix}=\begin{pmatrix}-16&10\\-36&28\end{pmatrix},$$

(h) $$\mathbf{AC}+\mathbf{BC}=\begin{pmatrix}-8&7\\-16&14\end{pmatrix}+\begin{pmatrix}-8&3\\-20&14\end{pmatrix}=\begin{pmatrix}-16&10\\-36&28\end{pmatrix}.$$

These special cases illustrate general properties of matrix manipulation,

Matrix multiplication is associative $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$ ((e) and (f)).

Matrix multiplication is distributive $(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}$ ((g) and (h))

Matrix multiplication is **not,** in general, commutative $\mathbf{BC}\neq\mathbf{CB}$ ((b) and (c)).

4. Matrix multiplication is only defined between matrices $\mathbf{A}$ and $\mathbf{B}$ if the number of columns in the matrix $\mathbf{A}$ equals the number of rows in the matrix $\mathbf{B}$. For example, if $\mathbf{A}$ is an $m\times p$ matrix and $\mathbf{B}$ is a $p\times n$ matrix, the matrix product is well-defined and $\mathbf{AB}$ is an $m\times n$ matrix. Note that $\mathbf{BA}$ is **not** well-defined unless $n=m$ in which case $\mathbf{BA}$ is a $p\times p$ matrix. $\mathbf{P}$ is a 2×4 matrix, $\mathbf{Q}$ is a 3×2 matrix, and $\mathbf{R}$ is a 3×3 matrix. Hence only $\mathbf{QP}$ (a 3×4 matrix), $\mathbf{RQ}$ (a 3×3 matrix), and $\mathbf{RR}=\mathbf{R}^2$ (a 3×3 matrix) are well-defined. All other combinations are meaningless. We find:

$$\mathbf{QP}=\begin{pmatrix}2&4\\1&-1\\3&-1\end{pmatrix}\begin{pmatrix}2&3&1&-4\\2&1&0&5\end{pmatrix}=\begin{pmatrix}2\cdot2+4\cdot2&2\cdot3+4\cdot1&2\cdot1+4\cdot0&2\cdot(-4)+4\cdot5\\1\cdot2+(-1)\cdot2&1\cdot3+(-1)\cdot1&1\cdot1+(-1)\cdot0&1\cdot(-4)+(-1)\cdot5\\3\cdot2+(-1)\cdot2&3\cdot3+(-1)\cdot1&3\cdot1+(-1)\cdot0&3\cdot(-4)+(-1)\cdot5\end{pmatrix}$$
$$=\begin{pmatrix}12&10&2&12\\0&2&1&-9\\4&8&3&-17\end{pmatrix}$$

$$\mathbf{RQ}=\begin{pmatrix}2&1&3\\4&-1&-2\\-1&0&1\end{pmatrix}\begin{pmatrix}2&4\\1&-1\\3&-1\end{pmatrix}=\begin{pmatrix}2\cdot2+1\cdot1+3\cdot3&2\cdot4+1\cdot(-1)+3\cdot(-1)\\4\cdot2+(-1)\cdot1+(-2)\cdot3&4\cdot4+(-1)\cdot(-1)+(-2)\cdot(-1)\\(-1)\cdot2+0\cdot1+1\cdot3&(-1)\cdot4+0\cdot(-1)+1\cdot(-1)\end{pmatrix}=\begin{pmatrix}14&4\\1&19\\1&-5\end{pmatrix}$$

$$\mathbf{R}^2=\begin{pmatrix}2&1&3\\4&-1&-2\\-1&0&1\end{pmatrix}\begin{pmatrix}2&1&3\\4&-1&-2\\-1&0&1\end{pmatrix}$$
$$=\begin{pmatrix}2\cdot2+1\cdot4+3\cdot(-1)&2\cdot1+1\cdot(-1)+3\cdot0&2\cdot3+1\cdot(-2)+3\cdot1\\4\cdot2+(-1)\cdot4+(-2)\cdot(-1)&4\cdot1+(-1)\cdot(-1)+(-2)\cdot0&4\cdot3+(-1)\cdot(-2)+(-2)\cdot1\\(-1)\cdot2+0\cdot4+1\cdot(-1)&(-1)\cdot1+0\cdot(-1)+1\cdot0&(-1)\cdot3+0\cdot(-2)+1\cdot1\end{pmatrix}$$
$$=\begin{pmatrix}5&1&7\\6&5&12\\-3&-1&-2\end{pmatrix}.$$

5. Define the inverse matrix $\mathbf{A}^{-1}=\begin{pmatrix}\underset{\sim}{a}_{11}&\underset{\sim}{a}_{12}\\\underset{\sim}{a}_{21}&\underset{\sim}{a}_{22}\end{pmatrix}$. This matrix must satisfy $\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$. The first equation implies $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}\Leftrightarrow\begin{pmatrix}9&6\\5&3\end{pmatrix}\begin{pmatrix}\underset{\sim}{a}_{11}&\underset{\sim}{a}_{12}\\\underset{\sim}{a}_{21}&\underset{\sim}{a}_{22}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ which is equivalent of a system of 4 equations with 4 unknowns (actually 2 × 2 Eqs. With 2 unknowns):

$$9\underset{\sim}{a}_{11}+6\underset{\sim}{a}_{21}=1$$
$$5\underset{\sim}{a}_{11}+3\underset{\sim}{a}_{21}=0$$
$$9\underset{\sim}{a}_{12}+6\underset{\sim}{a}_{22}=0$$ These are solve, for example by using Cramer's rule:
$$5\underset{\sim}{a}_{12}+3\underset{\sim}{a}_{22}=1$$

$$\underset{\sim}{a}_{11}=\frac{\begin{vmatrix}1&6\\0&3\end{vmatrix}}{\begin{vmatrix}9&6\\5&3\end{vmatrix}}=\frac{3}{-3}=-1;\ \underset{\sim}{a}_{21}=\frac{\begin{vmatrix}9&1\\5&0\end{vmatrix}}{\begin{vmatrix}9&6\\5&3\end{vmatrix}}=\frac{-5}{-3}=\frac{5}{3};\ \underset{\sim}{a}_{12}=\frac{\begin{vmatrix}0&6\\1&3\end{vmatrix}}{\begin{vmatrix}9&6\\5&3\end{vmatrix}}=\frac{-6}{-3}=2;\ \underset{\sim}{a}_{22}=\frac{\begin{vmatrix}9&0\\5&1\end{vmatrix}}{\begin{vmatrix}9&6\\5&3\end{vmatrix}}=\frac{9}{-3}=-3.$$

We check that $\begin{pmatrix} -1 & 2 \\ \frac{5}{3} & -3 \end{pmatrix}\begin{pmatrix} 9 & 6 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ \frac{5}{3} & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, confirming that indeed the inverse matrix is $\mathbf{A}^{-1} = \begin{pmatrix} -1 & 2 \\ \frac{5}{3} & -3 \end{pmatrix}$.

6. Applying the matrix $\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ to a vector $\mathbf{r}$ rotates it through an angle of θ counter-clock wise. Hence

$$\mathbf{Rr} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\cos\theta - 4\sin\theta \\ 3\sin\theta + 4\cos\theta \end{pmatrix} = \begin{pmatrix} \frac{3}{2} - 2\sqrt{3} \\ 3\frac{\sqrt{3}}{2} + 2 \end{pmatrix} \approx \begin{pmatrix} -1.964 \\ 4.598 \end{pmatrix} \text{ for } \theta = 60^\circ.$$

7. (a) $3\mathbf{M} = 3\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 12 & 3 \\ 9 & 6 \end{pmatrix}$, (b) $\det\mathbf{M} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 4\cdot 2 - 3\cdot 1 = 5$,

(c) $\det(3\mathbf{M}) = \begin{vmatrix} 12 & 3 \\ 9 & 6 \end{vmatrix} = 12\cdot 6 - 9\cdot 3 = 45 = 3^2\cdot 5$.

A determinant is multiplied by a factor r if all elements of one row (or column) are multiplied by r (see property 2, Fact Sheet 6). Therefore, if all elements of all n rows are multiplied by r, which is what happens if the parent matrix is multiplied by a factor r, the determinant will be multiplied by r^n, that is, if $\mathbf{A}$ is an $n\times n$ matrix, then $\det r\mathbf{A} = r^n \det\mathbf{A}$.