Fact Sheet 9 – Vectors II: Triple Products

Triple Scalar Product

- The triple scalar product, so called because three vectors are involved and the answer is a scalar quantity, is defined as

  $\mathbf{c}\bullet(\mathbf{a}\times\mathbf{b})$.

- In terms of components, this reads

$$\mathbf{c}\bullet(\mathbf{a}\times\mathbf{b}) = \mathbf{c}\bullet\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= c_x(a_y b_z - a_z b_y) - c_y(a_x b_z - a_z b_x) + c_z(a_x b_y - a_y b_x)$$

$$= \begin{vmatrix} c_x & a_x & b_x \\ c_y & a_y & b_y \\ c_z & a_z & b_z \end{vmatrix} = \det(\mathbf{c},\mathbf{a},\mathbf{b})$$

  that is, the determinant of the 3×3 matrix whit columns $\mathbf{c},\mathbf{a},$ and $\mathbf{b}$, respectively

- Note that $\begin{vmatrix} c_x & a_x & b_x \\ c_y & a_y & b_y \\ c_z & a_z & b_z \end{vmatrix} = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix} = \begin{vmatrix} b_x & c_x & a_x \\ b_y & c_y & a_y \\ b_z & c_z & a_z \end{vmatrix}$ since each successive determinant results from two column exchanges. Consequently,

  $\mathbf{c}\bullet(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\bullet(\mathbf{c}\times\mathbf{a}) = (\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = (\mathbf{b}\times\mathbf{c})\bullet\mathbf{a} = (\mathbf{c}\times\mathbf{a})\bullet\mathbf{b}$.

  so *all triple scalar products where the same vectors are rotated in cyclic order are equal*. This is irrespective of the order of the dot and cross product.

  The sign reverses when the vectors are not in cyclic order although the magnitude is unchanged. So $\mathbf{c}\bullet(\mathbf{a}\times\mathbf{b}) = -\mathbf{c}\bullet(\mathbf{b}\times\mathbf{a})$ but $|\mathbf{c}\bullet(\mathbf{a}\times\mathbf{b})| = |\mathbf{c}\bullet(\mathbf{b}\times\mathbf{a})|$.

- The *magnitude* $|\det(\mathbf{c},\mathbf{a},\mathbf{b})|$ of the triple scalar product is the *volume* of a parallelepiped whose sides are the vectors **a,b**, and **c**.

Triple Vector Product

- The triple vector product, so called because three vectors are involved and the result is a vector quantity, is defined as

  $\mathbf{e} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

- Bearing in mind that $\mathbf{b}\times\mathbf{c}$ is perpendicular to the plane of **b** and **c**, and that **e** is perpendicular to the plane of **a** and **b**×**c,** one can deduce that **e** lies in the plane of **b** and **c**. It follows that **e** can be written as a linear combination of **b** and **c** so that there exists numbers c_2 and c_3 such that $\mathbf{e} = c_2\mathbf{b} + c_3\mathbf{c}$.

  It can be shown, with a large piece of paper and some patience, that $c_2 = \mathbf{a}\bullet\mathbf{c}$ and $c_3 = -\mathbf{a}\bullet\mathbf{b}$. Therefore, we have the very useful identity

  $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\bullet\mathbf{c}) - \mathbf{c}(\mathbf{a}\bullet\mathbf{b})$, commonly known as the "**b**ac - **c**ab" rule.