Fact Sheet 8 – Matrix Inversion

How to find the inverse of a matrix using a specific formula with a detailed example.

- Only non-singular square matrices have inverses, that is, square matrices $\mathbf{A}$ for which the determinant is non-zero: $\det \mathbf{A} \neq 0$.
- When $\mathbf{A}$ is invetible, the formula for the inverse is $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}\mathbf{A}}{\det \mathbf{A}}$ where $\operatorname{adj}\mathbf{A}$ is the so-called <u>*adjoint matrix*</u> of $\mathbf{A}$.
- The <u>*adjoint matrix*</u> of $\mathbf{A}$ is defined by $\operatorname{adj}\mathbf{A} = \mathbf{C}^t$, where $\mathbf{C}$ is the associated <u>*matrix of the cofactors*</u> of $\mathbf{A}$ i.e., the adjoint matrix of $\mathbf{A}$ is the transposed matrix of its cofactors.
- The matrix of the cofactors is the matrix obtained by replacing every element a_{ij} of $\mathbf{A}$ the associated cofactor $C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}$, where $\mathbf{A}_{ij}$ is the ij th minor of $\mathbf{A}$ (i.e., the $(n-1)\times(n-1)$ matrix obtained from $\mathbf{A}$ by removing the i th row and the j th column.
- For a 2×2 matrix $\mathbf{A}$ with $\det \mathbf{A} \neq 0$, the inverse

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}^t = \frac{1}{\det \mathbf{A}}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

The following example is from question 3 of the Problems for Lecture 12:

The task is to find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix}$. The <u>*determinant*</u> of $\mathbf{A}$ is easily shown to be 2, which confirms that $\mathbf{A}$ has an inverse. The <u>*adjoint matrix*</u> of $\mathbf{A}$ is

$$\boldsymbol{adj}\mathbf{A} = \mathbf{C}^t = \begin{pmatrix} \begin{vmatrix} 1 & -4 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ -1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}^t = \begin{pmatrix} -7 & 2 & -3 \\ -8 & 2 & -4 \\ -11 & 2 & -5 \end{pmatrix}^t = \begin{pmatrix} -7 & -8 & -11 \\ 2 & 2 & 2 \\ -3 & -4 & -5 \end{pmatrix} \text{ and}$$

the <u>*inverse matrix*</u> $\mathbf{A}^{-1}$ for $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}\mathbf{A}}{\det \mathbf{A}} = \begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix}.$$

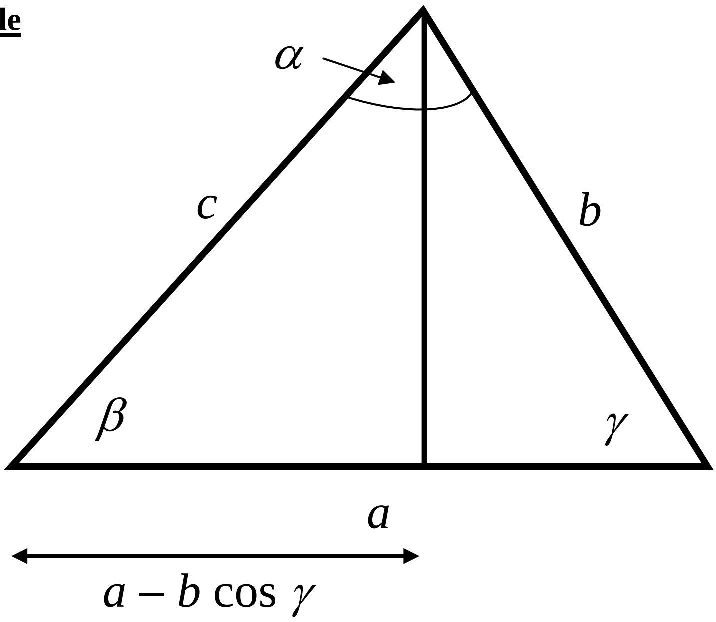
It is readily verified that

$$\begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -4 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{2} & -4 & -\frac{11}{2} \\ 1 & 1 & 1 \\ -\frac{3}{2} & -2 & -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Cosine and sine rules

Simple proofs of these well-known rules are given below.

The Cosine Rule



Using Pythagoras on the left-hand right-angled triangle yields

$$c^2 = (a - b\cos\gamma)^2 + (b\sin\gamma)^2 = a^2 + b^2(\cos^2\gamma + \sin^2\gamma) - 2ab\cos\gamma$$

leading to

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\gamma}$$

The Sine Rule

From the diagram, it is evident that

$$c\sin\beta = b\sin\gamma$$

and hence

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

The result is readily extended to include the third side and the third angle to read

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}$$