Fact Sheet 6 – Determinants

Let $\mathbf{A}$ be an $n \times n$ matrix where we denote the ij th element of the matrix $\mathbf{A}$ with a_{ij} The first index denotes the row number and the second index the column number. The *determinant of the square matrix* $\mathbf{A}$ is denoted by $\det \mathbf{A}$ or $|\mathbf{A}|$ and is defined inductively. Note that the determinant is defined <u>*only*</u> for a square matrix.

- The determinant of the 1×1 matrix $\mathbf{A} = (a_{11})$ is

$$\det \mathbf{A} = |a_{11}| = a_{11}. \tag{1}$$

- The determinant of the 2×2 matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}. \tag{2}$$

- To define the determinant of a $n \times n$ matrix, we introduce the so-called ij th minors $\mathbf{A}_{ij}$ of $\mathbf{A}$ which is the $(n-1)\times(n-1)$ matrix obtained from $\mathbf{A}$ by *deleting* the i th row and j th column.

- The determinant of the 3×3 matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \det \mathbf{A}_{11} - a_{12} \det \mathbf{A}_{12} + a_{13} \det \mathbf{A}_{13} \tag{3a}$$

Introducing explicitly the ij th minor, we find

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{3b}$$

and evaluating the 2×2 determinants using Eq.(2), we find

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \tag{3c}$$

Notice that every term in expansion of the determinant contains an element from each row and an element from each column.

- The determinant of the $n \times n$ matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{n1} & a_{n2} & \bullet & \bullet & a_{nn} \end{pmatrix}$ is defined by

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{n1} & a_{n2} & \bullet & \bullet & a_{nn} \end{vmatrix} = (-1)^{1+1} a_{11} \det \mathbf{A}_{11} + (-1)^{1+2} a_{12} \det \mathbf{A}_{12} + \cdots + (-1)^{1+n} a_{1n} \det \mathbf{A}_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \mathbf{A}_{1j}$$

In this definition, the determinate is expanded after the first row. However, one can show that the determinate may be expanded after any row or any column. That is

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij} \quad \text{(expanded after } i\text{th row)}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij} \quad \text{(expanded after } j\text{th column)}$$

Properties of Determinants

The evaluation of determinants can be simplified by using the following general properties:

1. The sign of a determinant is reversed if two rows (or two columns) are interchanged.
2. If all elements in any row (or column) are multiplied by a common factor r, the value of the determinant is multiplied by r.
3. The value of a determinant is zero if any row (or any column) is made up exclusively of zeros.
4. The value of a determinant is zero if two rows (or two columns) are identical.
5. The value of a determinant is zero if two rows (or two columns) are proportional.
6. The value of a determinant is unchanged if equal multiples of the elements of any row are added to the corresponding elements of any other row. The rule applies equally to columns.
7. The value of a determinant is unchanged if rows and columns are interchanged (i.e. if the underlying matrix is “transposed”).
8. If the elements of any row (or column) are the sums of two (or more) terms, the determinant can be written as the sum of two (or more) determinants. for example,

$$\begin{vmatrix} a_{11}+u & a_{12} & a_{13} \\ a_{21}+v & a_{22} & a_{23} \\ a_{31}+w & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} u & a_{12} & a_{13} \\ v & a_{22} & a_{23} \\ w & a_{32} & a_{33} \end{vmatrix}$$