Fact Sheet 5 – Linear Equations & Cramer's Rule

- A set of two *simultaneous linear equations* in two variables x_1, x_2 is on the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{22}x_1 + a_{22}x_2 = b_2 \end{array}\right\} \quad (1)$$

If $a_{11}a_{22} - a_{21}a_{12} \neq 0,$ they have a unique solution, namely

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \text{ and } x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}$$

(2)

- In matrix form, this set of linear equations reads as follows

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (1a)$$

Using determinants, the unique solution for *x* and *y* may be found using Cramer's rule:

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \text{ and } \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad (2a)$$

where we have defined a 2×2 determinant by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}\,. \quad (3)$$

- Note that the solutions for x_1 and x_2 have the same factor in the denominator, namely *the determinant of the coefficients* of the original equations. The determinant in the numerator for x_1 is obtained by changing the first column in the determinant of the coefficient with the right-hand side of Eq.(1a). Similarly, the determinant in the numerator for x_2 is obtained by changing the second column in the determinant of the coefficients with the right-hand side of Eq. (1a).

- Each member of Eq.(1) is the equation of a straight line. A unique solution to Eq.(1) exists, provided the lines are not parallel. *This is the case when the determinant of the coefficients is non-zero.* The unique solution given by Eq.(2) is the point where the lines cross. However, *in the case where the determinant of the coefficients is zero*, the lines are parallel. Then we have to distinguish between two different situations: (a) If the lines are parallel and have a point in common, the lines are identical and there are infinitely many solutions to Eq.(1). This happens is the two equations are proportional to one another. (b) If the lines are parallel and have no points in common there is no solution. The two equations are incompatible.

- A set of three simultaneous linear equations in three variables *x,y,z* is on the form

$$\left.\begin{aligned} a_1x+b_1y+c_1z=k_1 \\ a_2x+b_2y+c_2z=k_2 \\ a_3x+b_3y+c_3z=k_3 \end{aligned}\right\} \qquad (5)$$

In matrix form, this set of linear equations reads as follows

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad (5a)$$

If the determinant of the coefficients is non-zero, the unique solution for *x,y,z* may be found using Cramer's rule:

$$x_1=\frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_2=\frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_3=\frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

(6)

where we have defined a 3×3 determinant by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

. We say that the 3×3 determinant has been expanded after the first row.

Supplement: The following outlines the different situations that can occur for a set of three linear equations in three unknowns, when the matrix of the coefficients is "singular", that is, the determinant of the coefficients is zero. Under these circumstances, there is either no solution at all, or the solution is not uniquely defined – it may be a line, or a plane.

At the outset, we remind ourselves that:

- $ax + by + cz = k$ is the equation of a plane.
- The normal to the plane lies in the direction of the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- The perpendicular distance from the origin to the plane, that is, the distance in the direction of the normal is $\dfrac{k}{\sqrt{a^2 + b^2 + c^2}}$.
- In looking for the solution of three linear equations, we are concerned with the intersection of the planes. <u>Unless the determinant of the coefficients is zero, the three planes intersect at a point, and there is a unique solution</u>.
- When the determinant of the coefficients <u>*is*</u> zero, it means that the normals to the planes are coplanar, which is reflected in the fact that any row of the matrix is a linear superposition of the other two.

The six possible scenarios when the determinant is zero are as follows:

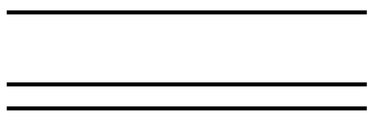
1. In scenario 1, all three planes are the same, so all three equations are the same, i.e., they are proportinal. But remember that they may not <u>*appear*</u> to be the same at first sight. An example is

$$x + 2y - 3z = 5$$
$$-2x - 4y + 6z = -10$$
$$3x + 6y - 9z = 15$$

Since all the equations are identical, any one equation defines the plane solution.

2. For scenario 2, the three planes are parallel, but all different. It follows that none of the planes intersects either of the others, so there is no solution. An example is

$$x + 2y - 3z = 5$$
$$x + 2y - 3z = 6$$
$$x + 2y - 3z = 8$$

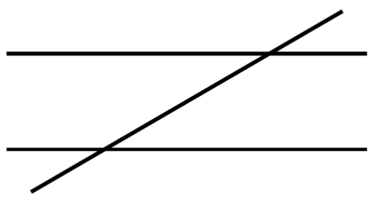


The left-hand sides are all the same, but the right-hand sides are different, making the equations contradictory and hence inconsistent.

There is of course an intermediate case between 1 and 2, in which two planes are coincident and the third is different. Still there is no solution.

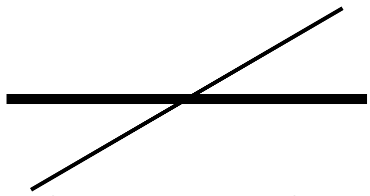
3. This scenario has two planes parallel, and the third inclined; again there is no solution. An example is

$$x + 2y - 3z = 5$$
$$x + 2y - 3z = 6$$
$$x + y + 3z = 4$$



4. Here the two parallel planes of the previous scenario are coincident, and there is a line solution. An example is

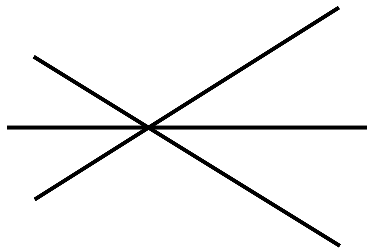
$$x + 2y - 3z = 5$$
$$-2x - 4y + 6z = -10$$
$$x + y + 3z = 4$$



Since the first two equations are identical, one can be discarded to find the equation of the line.

5. In this scenario, three different planes intersect on a common line like pages of a book. An example is

$$x + 2y - 3z = 5$$
$$x + y + 3z = 4$$
$$x + 3y - 9z = 6$$

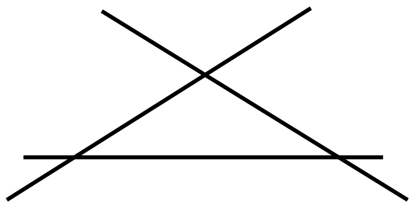


The third equation is twice the first minus the second.

Any two equations will yield the same line solution.

6. The previous scenario is a special case of the "toblerone" situation of this final scenario. Here, three planes intersect in three different parallel lines, and there is no solution of any kind. An example is

$$x + 2y - 3z = 5$$
$$x + y + 3z = 4$$
$$x + 3y - 9z = 7$$



The only difference between these equations and those in scenario 5 is that the number on the right-hand-side of the third equation has been changed.

<u>The homogeneous case</u>

A special case occurs when the right-hand sides of a set of linear equations are all zero. Such equations clearly have the trivial solution $x = y = z = 0$, indicating that the three planes cross at the origin. This is the *<u>only</u>* solution if the determinant of the coefficients is non-zero. However, if the determinant of the coefficients *<u>is</u>* zero, there will also be line (or plane) solutions, as in cases 4, 5 and 1 above. The line (or plane) solutions will of course pass through the origin in all cases.

Supplement: Systematic Elimination

It is easy to solve a set of two linear equations in two unknowns by eliminating one variable to obtain a solution for the other. This is then substituted back to obtain the solution for the first variable. For a set of *three* equations in *three* unknowns, you can use a similar procedure, but things are more complicated now, and it's desirable to have a proper systematic system. For n equations in n unknowns ($n > 3$), such a system is absolutely essential!

The example of systematic elimination that will be presented in Lecture 8 is reproduced below. At the outset, remember the "Rules of the Game":

The Rules of the Game

The solution of a set of linear equations is not changed by any of the following operations:

(i) Changing the order of the equations.

(ii) Multiplying all terms in any equation by the same non-zero constant.

(iii) Adding a multiple of any equation to any other equation. The multiple can be negative (so addition includes subtraction), and it need not be an integer multiple.

The strategy will be to use these principles to leave only z in the last equation, only y and z in the next last, and so on.

The three equations in our example were

$$x + 3y + z = 8$$
$$2x + y + 3z = 7$$
$$x + y - z = 2$$

Step 1: Subtract the first equation from the third, and twice the first equation from the second to eliminate x in the second and third equations:

$$x + 3y + z = 8$$
$$0 - 5y + z = -9$$
$$0 - 2y - 2z = -6$$

Notice the two zeros, which it was the purpose of step 1 to create!

Step 2: Now subtract 2/5 times the second equation in the new set from the third to eliminate y in the third equation. (You may also have added 3/5 to the first equation to eliminate y in the first equation.)

$$x + 3y + z = 8$$
$$0 - 5y + z = -9$$
$$0 + 0 - \tfrac{12}{5} z = -\tfrac{12}{5}$$

Notice that there is now a third zero, indeed the bottom left section of the three equations is now all zero, which is the whole object of the exercise!

The solution can now be read straight off. The third equation yields $z = 1$. Inserting $z=1$ into the second yields $y=2$. Substituting $(y,z)=(2,1)$ into the first equation yields $x=1$. Hence the unique solution exiats and it is given by $(x,y,z)=(1,2,1)$. That's all there is to it!