Fact Sheet 4 – Directions, Lines and Planes(draft version)

- A vector $\mathbf{d}=(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ defines a particular direction in $\mathbb{R}^3$. The coefficients $x, y,$ and z are called the <u>*direction ratios*</u>. With $|\mathbf{d}|=\sqrt{x^2+y^2+z^2}$ a unit vector $\hat{\mathbf{d}}$ defining a particular direction in $\mathbb{R}^3$ may be written

  $$\hat{\mathbf{d}}=\frac{\mathbf{d}}{|\mathbf{d}|}=\frac{x}{|\mathbf{d}|}\mathbf{i}+\frac{y}{|\mathbf{d}|}\mathbf{j}+\frac{z}{|\mathbf{d}|}\mathbf{k}=\cos\alpha\,\mathbf{i}+\cos\beta\,\mathbf{j}+\cos\gamma\,\mathbf{k}=\ell\,\mathbf{i}+m\,\mathbf{j}+n\,\mathbf{k}$$

  where $\cos\alpha, \cos\beta,$ and $\cos\gamma$ or $\ell, m,$ and n are the <u>*direction cosines*</u> of $\hat{\mathbf{d}}$ as indicated. Since $\hat{\mathbf{d}}$ is a unit vector, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ or, equivalently, $\ell^2+m^2+n^2=1$.
  Example: To find the direction cosines of a vector $\mathbf{d}$ that is <u>not</u> a unit vector, divide by its magnitude $|\mathbf{d}|$ to obtain a unit vector $\hat{\mathbf{d}}$ in the same direction. For example, for the vector $\mathbf{d}=2\mathbf{i}+4\mathbf{j}+3\mathbf{k}$, divide by $|\mathbf{d}|=\sqrt{x^2+y^2+z^2}=\sqrt{29}$ to obtain the unit vector $\hat{\mathbf{d}}=\frac{2}{\sqrt{29}}\mathbf{i}+\frac{4}{\sqrt{29}}\mathbf{j}+\frac{3}{\sqrt{29}}\mathbf{k}$. The coefficients of $\mathbf{d}$ (2,4,3) are the <u>*direction ratios*</u>, and the coefficients $\left(\frac{2}{\sqrt{29}},\frac{4}{\sqrt{29}},\frac{3}{\sqrt{29}}\right)$ of $\hat{\mathbf{d}}$ are the <u>*direction cosines*</u> .

- The angle θ between two unit vectors can be found using the dot(scalar)-product

  $$\cos\theta=\hat{\mathbf{d}}_{\mathbf{1}}\bullet\hat{\mathbf{d}}_{\mathbf{2}}=\ell_1\ell_2+m_1m_2+n_1n_2\,.$$

  The angle θ between two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ with $\hat{\mathbf{a}}=\frac{\mathbf{a}}{|\mathbf{a}|}$, $\hat{\mathbf{b}}=\frac{\mathbf{b}}{|\mathbf{b}|}$ follows from

  $$\cos\theta=\hat{\mathbf{a}}\bullet\hat{\mathbf{b}}=\frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\frac{a_xb_x+a_yb_y+a_zb_z}{|\mathbf{a}||\mathbf{b}|}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}}\,.$$

- A line passing through a point R_0 with position vector $\mathbf{r}_0$ in a direction $\hat{\mathbf{d}}$ is given by $\mathbf{r}=\mathbf{r_0}+\lambda\hat{\mathbf{d}},\ \lambda\in\mathbb{R}$. This parametric equation for a line is valid in $\mathbb{R}^n$. On component form $x_i=x_{0i}+\lambda d_i, i=1,2,\ldots n.$ Isolating λ we find $\lambda=\frac{x_1-x_{01}}{d_1}=\cdots=\frac{x_n-x_{0n}}{d_n}$.

  Example: Consider $\mathbb{R}^2$ with $\mathbf{r}_0=(x_0,y_0)$ and $\mathbf{d}=(d_{x,}d_y)$. Then the parametric equation on component form is

  $x=x_0+\lambda d_x$ and $y=y_0+\lambda d_y$. Isolating λ we find $\lambda=\frac{x-x_0}{d_x}=\frac{y-y_0}{d_y}$ or, equivalently, $y-y_0=\frac{d_y}{d_x}(x-x_0)$ where $\frac{d_y}{d_x}$ is the slope of the line.

- The latter form is equivalent with the "well-know" equation of <u>*a straight line in 2D*</u>. A line with gradient α and y-axis intercept y_0 (i.e. $x_0=0$) is given by

  $$y=y_0+\alpha x\,.$$

Similarly, a line passing through (x_0, y_0) in a direction of a unit vector $\hat{\mathbf{d}} = (\ell, m)$ defined by ℓ and m is given by $\frac{x - x_0}{\ell} = \frac{y - y_0}{m}$.

A straight through a point $\mathbf{a}$ and with a normal vector $\mathbf{n} = (a, b)$ (the notation is unfortunate – please do *not* identify a with $|\mathbf{a}|$ ☺) must satisfy the equation $(\mathbf{r} - \mathbf{a})\bullet\mathbf{n} = 0 \Leftrightarrow \mathbf{r}\bullet\mathbf{n} = \mathbf{a}\bullet\mathbf{n}$. Denoting $\mathbf{a}\bullet\mathbf{n} = k,$ the line can also be specified in the form

$$ax + by = k$$

where a and b are the <u>*direction ratios*</u> of the normal vector $\mathbf{n}$, i.e. the vector drawn perpendicular to the line from the origin. Note that $\mathbf{a}\bullet\mathbf{n}$ is the projection of $\mathbf{a}$ onto $\mathbf{n}$. Hence $k = |\mathbf{n}| \cdot$ perpendicular distance from the origin to the line $\equiv |\mathbf{n}|\, p$. Dividing all terms in the equation by $|\mathbf{n}| = \sqrt{a^2 + b^2}$, one obtains

$$\underbrace{\frac{a}{\sqrt{a^2 + b^2}}}_{\ell'} x + \underbrace{\frac{b}{\sqrt{a^2 + b^2}}}_{m'} y = \frac{k}{\sqrt{a^2 + b^2}} = p$$

where the coefficients (ℓ' and m') of x and y are now the <u>*direction cosines*</u> of the normal vector, and p on the right hand side is the <u>*length*</u> of the normal, i.e. the perpendicular distance from the origin to the line. Since the normal is (by definition) perpendicular to the line, it follows that

$$(\ell, m)\bullet(\ell', m') = \ell\ell' + mm' = 0$$

- The <u>*equation of a plane*</u> in 3D through a point A with position vector $\mathbf{a}$ and perpendicular to a normal vector $\mathbf{n}$ is

  $(\mathbf{r} - \mathbf{a})\bullet\mathbf{n} = 0 \Leftrightarrow \mathbf{r}\bullet\mathbf{n} = \mathbf{a}\bullet\mathbf{n}$. If $\mathbf{n} = (a, b, c)$ and $\mathbf{a}\bullet\mathbf{n} = k$ we find

  $$ax + by + cz = k\,.$$

  Dividing through by the magnitude of the normal vector $|\mathbf{n}| = \sqrt{a^2 + b^2 + c^2}$ yields

  $$\underbrace{\frac{a}{\sqrt{a^2 + b^2 + c^2}}}_{\ell'} x + \underbrace{\frac{b}{\sqrt{a^2 + b^2 + c^2}}}_{m'} y + \underbrace{\frac{c}{\sqrt{a^2 + b^2 + c^2}}}_{n'} z = \frac{k}{\sqrt{a^2 + b^2 + c^2}} = p$$

  where, by analogy with the case of a line in 2D above, ℓ', m' and n' are the direction cosines of the normal vector, i.e. the vector drawn normal to the plane from the origin. The parameter p is the length of the normal, i.e. the perpendicular distance from the origin to the plane.

Supplement – Planes

This sheet contains key information about the equations of planes.

<u>The equation of a plane</u>

The equation of a plane is

$$ax + by + cz = k \tag{1}$$

If you're unsure why this is the equation of a plane, notice that if two of the variables are specified (say *x* and *y*), the equation fixes the third (*z* in this case). So the equation clearly specifies a surface, and the linearity of the equation ensures that it is a <u>*plane*</u> surface. Notice also that if you take a section through the surface (say *y* = 0, which will give you the intersection of the specified plane with the *x*-*z* plane), the resulting equation ($ax + cz = k$) is the equation of a straight line.

<u>Normal to a plane</u>

The vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is normal to the plane. Dividing through by $|\mathbf{n}| = \sqrt{a^2 + b^2 + c^2}$ gives the associated <u>*unit*</u> vector as

$$\hat{\mathbf{n}} = \underbrace{\frac{a}{\sqrt{a^2 + b^2 + c^2}}}_{\ell'}\mathbf{i} + \underbrace{\frac{b}{\sqrt{a^2 + b^2 + c^2}}}_{m'}\mathbf{j} + \underbrace{\frac{c}{\sqrt{a^2 + b^2 + c^2}}}_{n'}\mathbf{k} \tag{2}$$

where ℓ', m' and n' are the direction cosines of the normal to the plane.

<u>Vector equation of a plane</u>

A plane can also be defined by the vector equation

$$\mathbf{r} \bullet \hat{\mathbf{n}} = p \tag{3}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and *p* is the perpendicular distance from the origin to the plane. Think of eq.(3) as defining the locus of all points **r** that lie on the plane.

Expressed in terms of components, eq.(3) becomes

$$\ell' x + m' y + n' z = p \tag{4}$$

because $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\hat{\mathbf{n}} = \ell'\mathbf{i} + m'\mathbf{j} + n'\mathbf{k}$. Eq.(4) is very similar to Eq.(1), indeed Eq.(1) can be converted to this form simply by dividing all terms by $|\mathbf{n}| = \sqrt{a^2 + b^2 + c^2}$ to obtain

$$\underbrace{\frac{a}{\sqrt{a^2 + b^2 + c^2}}}_{\ell'}x + \underbrace{\frac{b}{\sqrt{a^2 + b^2 + c^2}}}_{m'}y + \underbrace{\frac{c}{\sqrt{a^2 + b^2 + c^2}}}_{n'}z = \frac{k}{\sqrt{a^2 + b^2 + c^2}} = p\,. \tag{5}$$

An example

Consider the plane

$2x + 6y - 3z = 14$

From the information provided above, the following properties can immediately be deduced:

- A normal vector to the plane is $\mathbf{n} = (2, 6, -3)$ (or any multiple hereof, i.e. $\alpha\mathbf{n}, \alpha \neq 0$).
- The perpendicular distance from the origin to the plane is
  $$p = \frac{14}{\sqrt{2^2 + 6^2 + 3^2}} = \frac{14}{7} = 2 .$$
- The unit normal to the plane is
  $\hat{\mathbf{n}} = \frac{2}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{3}{7}\mathbf{k} = \left(\frac{2}{7}, \frac{6}{7}, -\frac{3}{7}\right)$ the direction cosines of which are

  $\ell' = 2/7, m' = 6/7, n' = -3/7$.

Note that the position vector (with respect to the origin) of the nearest point on the plane is

$$p\hat{\mathbf{n}} = \frac{4}{7}\mathbf{i} + \frac{12}{7}\mathbf{j} - \frac{6}{7}\mathbf{k} = \left(\frac{4}{7}, \frac{12}{7}, -\frac{6}{7}\right)$$

It's easy to verify that the coordinates of $p\hat{\mathbf{n}}$ do indeed satisfy the equation $2x + 6y - 3z = 14$. Note that, of course, $p\hat{\mathbf{n}}$ is a multiple of $\mathbf{n}$.

What happens if the right-hand side of the equation is negative? Suppose, for example, the equation had been

$2x + 6y - 3z = -14$. Then you should simply multiply the equation with -1 to obtain

$-2x - 6y + 3z = 14$. In this case, the plane lies on the opposite side of the origin and $\hat{\mathbf{n}}$ is now

$\hat{\mathbf{n}} = -\frac{2}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$. Still $p = 2$ which leads to $p\hat{\mathbf{n}} = -\frac{4}{7}\mathbf{i} - \frac{12}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$.