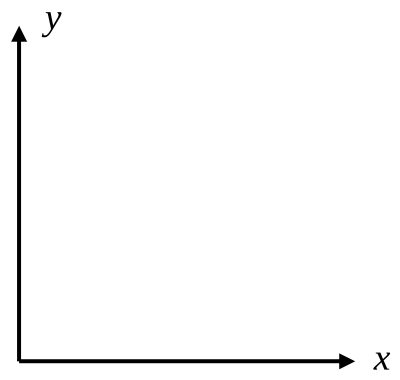
Fact Sheet 3 – Vectors I

Coordinate System

- A <u>*right-handed Cartesian coordinate system*</u> is one in which perpendicular *x*, *y*, and *z* axes are oriented such that rotating the axes from *x* towards *y* about the *z*-axis would cause a right-handed thread to advance in the +*z* direction.

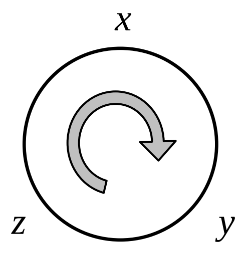
  For the axes shown below, the rotation defined above is counter-clockwise and hence, to form a right-handed system, the *z*-axis must come **<u>OUT OF THE PAPER</u>**.



- Cyclic order for the *xyz* coordinates means any sequence of three taken from *x y z x y z* ....

  Hence *xyz*, *yzx*, and *zxy* are in cyclic order, whereas *xzy*, *yxz* and *zyx* are not.

- The three cyclic order sequences can be obtained from the diagram below



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ADDITIONAL MATERIAL (USEFUL FOR LATER)

The extension of these ideas to sequences of numbers comes in useful when evaluating determinants

- Sequences of three numbers may be called <u>*even*</u> when they are in cyclic order, and <u>*odd*</u> when they are not. The three even sequences are therefore

  123 231 312

- Sequences of four (or more) numbers can also be classified as <u>*even*</u> or <u>*odd*</u>. Even sequences are those that can be converted to the basic sequence 1234 by an even number of exchanges. Odd sequences are defined similarly.

  Typical even sequences are: 2143 4132 3241

  Typical odd sequences are: 3421 1432 2134

Vector Basics

- A <u>*vector*</u> has magnitude and direction, whereas a <u>*scalar*</u> quantity has only magnitude.

- Vectors are written either in boldface type (e.g. **A**) or (esp. on the board) underlined (e.g. <u>*A*</u>).

- The component of a vector along one of the coordinate axes (say *x*) is written A_x.

- The <u>*magnitude*</u> of a vector, associated with the length of the arrow in a diagrammatic representation, is written $|\mathbf{A}|$ or sometimes just *A*.

  Hence $|\mathbf{A}| \equiv A = \sqrt{A_x^2 + A_y^2}$ in 2D, or $|\mathbf{A}| \equiv A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ in 3D.

- A vector of unit length (magnitude 1) is called a <u>*unit vector*</u>. The unit vector $\hat{\mathbf{a}}$ in the direction of a vector **A** is obtained by dividing the vector by its magnitude $|\mathbf{A}|$ so that

  $$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

  A circumflex is frequently used to denote a unit vector.

- The symbols **i**, **j**, and **k** are commonly used to represent unit vectors along the *x*, *y*, and *z* axes. Circumflexes are normally <u>*not*</u> applied for these special unit vectors.

- Using **i**, **j**, and **k**, a general vector **A** can be written

  $$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

- When a vector is used to define position relative to the origin, it is called a <u>*position vector*</u>. A point P with coordinates (*x*, *y*, *z*) is duly defined by the vector

  $$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

  Since $|\mathbf{r}| \equiv r = \sqrt{x^2 + y^2 + z^2}$, the unit vector in the direction of **r** is

  $$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} \equiv \frac{\mathbf{r}}{r} = \frac{\mathbf{r}}{\sqrt{x^2 + y^2 + z^2}}$$

- Two vectors are identical if they have the same magnitude and direction. Line of action is not part of the definition of a vector.

- Multiplying a vector by a scalar factor is straightforward. For example, the vector 3**A** has three times the magnitude of **A** and points in the same direction. In terms of components, $3\mathbf{A} = (3A_x, 3A_y, 3A_z)$. Similarly the vector $-\mathbf{A}$ has the same magnitude as **A** but points in the reverse direction; its components are $(-A_x, -A_y, -A_z)$.

- Vectors **A** and **B** are added simply by adding their components so that

  $$\mathbf{C} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

  Hence $C_x = A_x + B_x$ etc. Geometrically one obtains the resultant vector **C** by placing the tail of **B** at the head of **A** (or vice versa) and completing the triangle.

Vector Multiplication

There are two types of vector multiplication:

- The *scalar product*, so called because the results is a *scalar* quantity (and also called the *dot product* because a dot is the symbol used) is defined as

  $$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$$

  where $|\mathbf{A}|$ and $|\mathbf{B}|$ are the magnitudes of the two vectors and θ is the angle between the vectors. In terms of components,

  $$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

- The vector *product*, so called because the result is a *vector* quantity (and also called the *cross product* because a cross is the symbol commonly used) is defined as

  $\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\, \hat{\mathbf{n}}$ (some books use the symbolism $\mathbf{A} \wedge \mathbf{B}$ for the vector product)

  where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of **A** and **B**, directed such that **A**, **B**, and $\hat{\mathbf{n}}$ form a right-handed system (see Fact Sheet 3). In terms of components

  $$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

  In determinant form (see later in the course), the vector product can be written

  $$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- From the definitions, it is easy to see that, whereas the scalar product is commutative, the vector product is non-commutative i.e.

  $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$ but $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

- The unit vectors **i**, **j**, and **k** obey the following fairly obvious scalar product rules:

  $$\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$$

  but the scalar product of all *unlike* pairs is zero, i.e.

  $$\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$$

- The vector product of any vector with itself is always zero so

  $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$. Indeed, the vector product of two parallel vectors equals zero.

  Provided a right-handed system of axes is in use

  $\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$

  $\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

  Note that the sign is positive when the three unit vectors are in *cyclic order* (i j k i j k ....), and negative otherwise (see Fact Sheet 3).

- Note that right-handedness occurs both in the definition of the vector product, and in the definition of a right-handed set of *xyz* axes. Failure to adhere strictly to these rules will lead to sign errors.

Supplement – Additional material on the cross product

The cross product of two vectors is defined by

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\,\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of **A** and **B**, directed such that **A**, **B**, and $\hat{\mathbf{n}}$ form a right-handed system (see Fact Sheet 3).

In particular, it follows that,

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \quad \mathbf{j}\times\mathbf{k}=\mathbf{i} \quad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

$$\mathbf{j}\times\mathbf{i}=-\mathbf{k} \quad \mathbf{k}\times\mathbf{j}=-\mathbf{i} \quad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$

These results apply in a right-handed coordinate system, which should always be used.

We have shown that, in component form,

$$\boxed{\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}}$$

The proof of this result is as follows:

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\times(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})\\ &= A_xB_x\mathbf{i}\times\mathbf{i} + A_yB_y\mathbf{j}\times\mathbf{j} + A_zB_z\mathbf{k}\times\mathbf{k}\\ &+ A_xB_y\mathbf{i}\times\mathbf{j} + A_yB_x\mathbf{j}\times\mathbf{i}\\ &+ A_yB_z\mathbf{j}\times\mathbf{k} + A_zB_y\mathbf{k}\times\mathbf{j}\\ &+ A_zB_x\mathbf{k}\times\mathbf{i} + A_xB_z\mathbf{i}\times\mathbf{k}\end{aligned}$$

The first three terms are zero because $\mathbf{i}\times\mathbf{i}=0$ etc. Using the fact that $\mathbf{i}\times\mathbf{j}=-\mathbf{j}\times\mathbf{i}$ etc, leads to the result in the box above. The result can be written in determinant form (see later in the course) as

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$