Fact Sheet 2 – Complex Numbers I

- A *complex number* is one that involves the factor $i = \sqrt{-1}$. Note that j is sometimes used instead of i in certain applications, for example in electricity, where i could be confused with current.
- A general complex number z can be written in the standard form $z = x + iy$, where x and y are known respectively as the *real part* and the *imaginary part* of z, that is, Re(z) = x and Im(z) = y. Needless to say, x is no more real, and no less imaginary, than y.
- A complex number is represented geometrically by a point in the *complex plane* (or *Argand diagram*) in which the real part is plotted on the x-axis and the imaginary part on the y-axis.
- Equally, one can use polar coordinates and write

  $z = r(\cos\theta + i\sin\theta)$

  where $r\cos\theta$ is the real part and $r\sin\theta$ is the imaginary part of z. The parameter r is called the *modulus* of z (basically its magnitude); it is written $|z|$. The angle θ is called the *argument* or *phase* of z.

  Clearly $|z| \equiv r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$.
- It is easy (as in the lectures) to show that

  $$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

  The late Professor Richard Feynmann said this was "*the most remarkable formula in mathematics*", linking trigonometry on the lhs with algebra on the rhs. It follows that

  $$\boxed{z = re^{i\theta}}$$
- The *complex conjugate* of a complex number is obtained by reversing the sign of the imaginary part, which can be done by replacing i with $-i$ everywhere it appears. Thus, if $z = x + iy = r\,\mathrm{e}^{i\theta}$, then $z^* = x - iy = r\,\mathrm{e}^{-i\theta}$. Note the use of * to indicate complex conjugation.
- Complex numbers can be added, subtracted, multiplied, and divided in the normal way using either the (x, y) or the (r, θ) form. Just remember to keep track of the factor i. Hence, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

  The square of i can of course be replaced by -1 so, for example, $i^3 \equiv i \times i^2 = -i$ etc. Hence $z^2 = (x^2 - y^2) + 2ixy = r^2 e^{2i\theta}$ and $zz^* \equiv |z|^2 = r^2 = x^2 + y^2$. Notice that multiplying a complex number by its complex conjugate yields the square of the modulus.

  A useful trick to facilitate the division of one complex number by another is to multiply top and bottom by the complex conjugate of the denominator e.g.

  $$\frac{4+5i}{2-3i} = \frac{4+5i}{2-3i} \times \frac{2+3i}{2+3i} = \frac{-7+22i}{13} = -\frac{7}{13} + i\frac{22}{13}.$$