Fact Sheet 13 – 2nd Order ODEs

Undamped Equations

(1) $\boxed{\ddot{x} = \alpha^2 x}$

The trial solution $x(t) = Ae^{ut}$ yields $u^2 x = \alpha^2 x \Leftrightarrow u^2 = \alpha^2 \Leftrightarrow u = \pm\alpha$, so the general solution is $x(t) = Ae^{+\alpha t} + Be^{-\alpha t}$.

The two adjustable constants A and B must be chosen to match the initial conditions at $t = 0$.

If $x(0) \equiv x_0 = A + B$ and $\dot{x}(0) \equiv v_0 = \alpha(A - B)$, it follows that $A = \frac{1}{2}\left(x_0 + \frac{v_0}{\alpha}\right)$ and $B = \frac{1}{2}\left(x_0 - \frac{v_0}{\alpha}\right)$.

This leads to $x(t) = x_0 \frac{1}{2}\left(e^{+\alpha t} + e^{-\alpha t}\right) + \frac{v_0}{\alpha}\frac{1}{2}\left(e^{+\alpha t} - e^{-\alpha t}\right)$, and therefore

$$\boxed{x(t) = x_0 \cosh \alpha t + \frac{v_0}{\alpha} \sinh \alpha t}$$

(2) $\boxed{\ddot{x} = -\omega_0^2 x}$

The trial solution $x(t) = \tilde{a}e^{ut}$ yields $u^2 x = -\omega_0^2 x \Leftrightarrow u^2 = -\omega_0^2 \Leftrightarrow u = \pm i\omega_0$, so the general solution is $x(t) = \tilde{a}e^{+i\omega_0 t} + \tilde{b}e^{-i\omega_0 t}$. Since $x(t)$ is real, we have $\tilde{b} = \tilde{a}^*$. Writing $\tilde{a} = ae^{i\phi}$ then we have $x(t) = ae^{i\phi}e^{+i\omega_0 t} + ae^{-i\phi}e^{-i\omega_0 t} = a\left(e^{+i(\omega_0 t+\phi)} + e^{-i(\omega_0 t+\phi)}\right)$ which leads to the general solution

$$\boxed{x(t) = 2a\cos(\omega_0 t + \phi) = A\cos(\omega_0 t + \phi)}$$

The two adjustable constants A and ϕ must be chosen to match the initial conditions.

Damped Equations

(3) $\boxed{\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = 0}$

The parameter Γ represents friction or resistance. When $\Gamma = 0$, we recover the undamped Eq. (2).

The trial solution $x(t) = Ae^{ut}$ yields $u^2 + 2\Gamma u + \omega_0^2 = 0 \Leftrightarrow u = \dfrac{-2\Gamma \pm \sqrt{4\Gamma^2 - 4\omega_0^2}}{2} = -\left(\Gamma \pm \sqrt{\Gamma^2 - \omega_0^2}\right)$.

Three separate cases need to be considered depending on the value of $\Gamma^2 - \omega_0^2$ which determines whether the equation for *u* has two real solutions ($\Gamma^2 > \omega_0^2$), one real (double) solution ($\Gamma^2 = \omega_0^2$), or two imaginary solutions ($\Gamma^2 < \omega_0^2$). The three qualitatively different cases are commonly known as

$$\begin{cases} \Gamma^2 > \omega_0^2 & \text{Overdamped Motion,} \\ \Gamma^2 = \omega_0^2 & \text{Critically damped Motion,} \\ \Gamma^2 < \omega_0^2 & \text{Underdamped Motion.} \end{cases}$$

Overdamped Motion $\left(\Gamma^2 > \omega_0^2 \Leftrightarrow \Gamma > \omega_0\right)$

In this case $\Gamma^2 - \omega_0^2 > 0$, so $u = -(\Gamma \pm \sqrt{\Gamma^2 - \omega_0^2}) = -\mu_\pm$ where we define the two *real* numbers $\mu_+ = \Gamma + \sqrt{\Gamma^2 - \omega_0^2}$ and $\mu_- = \Gamma - \sqrt{\Gamma^2 - \omega_0^2}$. Therefore, the general solution is

$$\boxed{x(t) = Ae^{-\mu_- t} + Be^{-\mu_+ t}}$$

The two adjustable constants A and B must be chosen to match the initial conditions.

Underdamped Motion $\left(\Gamma^2 < \omega_0^2 \Leftrightarrow \Gamma < \omega_0\right)$.

In this case $\Gamma^2 - \omega_0^2 < 0$, so $u = -\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2} = -\Gamma \pm i\omega_0'$ where we define $\omega_0' = \sqrt{\omega_0^2 - \Gamma^2}$.

Therefore,

$x(t) = e^{-\Gamma t}\left(\tilde{a}e^{+i\omega_0' t} + \tilde{b}e^{-i\omega_0' t}\right)$. Since $x(t)$ is real, we have $\tilde{b} = \tilde{a}^*$. Writing $\tilde{a} = ae^{i\phi}$, we find $x(t) = e^{-\Gamma t}\left(ae^{i\phi}e^{+i\omega_0' t} + ae^{-i\phi}e^{-i\omega_0' t}\right) = e^{-\Gamma t}\left(a\left[e^{+i(\omega_0' t+\phi)} + e^{-i(\omega_0' t+\phi)}\right]\right)$ so the general solution is

$$\boxed{x(t) = 2ae^{-\Gamma t}\cos(\omega_0' t + \phi) = Ae^{-\Gamma t}\cos(\omega_0' t + \phi)}$$

The two adjustable constants A and ϕ must be chosen to match the initial conditions. Note that with $\Gamma = 0$ we recover the solution to Eq.(2).

Critical Damped Motion $\left(\Gamma^2 = \omega_0^2 \Leftrightarrow \Gamma = \omega_0\right)$

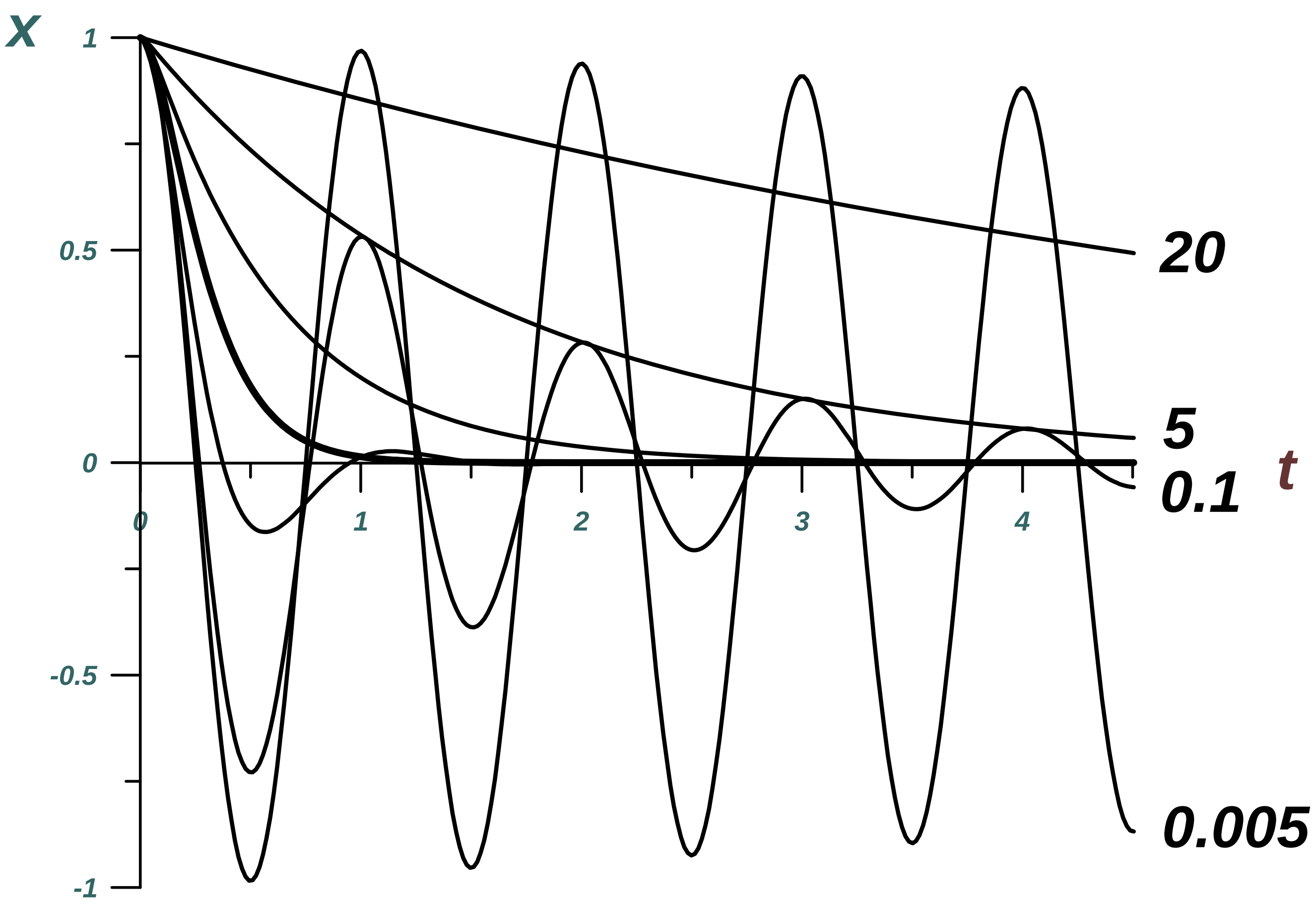
In this case, $u = -\Gamma$ and the only solution would appear to be $x(t) = Ae^{-\Gamma t}$; but this is unsatisfactory because it has only one adjustable constant *A*. The general solution is actually

$$\boxed{x(t) = (A + Bt)e^{-\Gamma t}}$$

See Problem Sheet 17 question 4 for a proof of this result.

The two adjustable constants A and B must be chosen to match the initial conditions.

The picture below shows graphs of the solutions for values of Γ / ω_0 corresponding to all three cases. The time axis is in units of $2\pi / \omega_0$. Results are displayed for $\Gamma / \omega_0 = 20,\ 5, 2, 1, 0.5, 0.1,$ and 0.005.



x
1
0.5
0
-0.5
-1
0
1
2
3
4
t
20
5
0.1
0.005

Driven (or Forced) Oscillation and Resonance

(4) $\boxed{\ddot{x}+2\Gamma\dot{x}+\omega_0^2 x = F\cos\omega t}$

The term $F\cos\omega t$ represents a driving force. When $F=0$, we recover the damped Eq. (3).

Let $x(t)=\mathrm{Re}\left(\tilde{x}(t)\right)$ and construct the analogous complex differential equation namely

$$\boxed{\ddot{\tilde{x}}+2\Gamma\dot{\tilde{x}}+\omega_0^2\tilde{x}=Fe^{i\omega t}\,.}$$

Let $\tilde{x}(t)=\tilde{A}e^{i\omega t}$ denote a trial solution. Substituting $\tilde{x}(t)$ into the complex differential eq., we find $-\omega^2\tilde{A}e^{i\omega t}+2\Gamma i\omega\tilde{A}e^{i\omega t}+\omega_0^2\tilde{A}e^{i\omega t}=Fe^{i\omega t}$, that is, $\left(-\omega^2+2\Gamma i\omega+\omega_0^2\right)\tilde{A}=F$ which yields an eq. for $\tilde{A}$: $\tilde{A}=\dfrac{F}{\omega_0^2-\omega^2+2i\omega\Gamma}$. Therefore $\tilde{x}(t)=\dfrac{F}{\omega_0^2-\omega^2+2i\omega\Gamma}e^{i\omega t}$. In order to facilitate taking the real part to find $x(t)$, we write $\omega_0^2-\omega^2+2i\omega\Gamma=\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\omega\Gamma\right)^2}\,e^{i\delta}$, where $\tan\delta=\dfrac{2\omega\Gamma}{\omega_0^2-\omega^2}$. Then

$\tilde{x}(t)=\dfrac{F}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\omega\Gamma\right)^2}}e^{i(\omega t-\delta)}$ and it follows that the general solution to Eq.(4) is

$$\boxed{x(t)=\mathrm{Re}\left(\tilde{x}(t)\right)=\frac{F}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\omega\Gamma\right)^2}}\cos\left(\omega t-\delta\right)\text{, where }\tan\delta=\frac{2\omega\Gamma}{\omega_0^2-\omega^2}\,.}$$

Three special cases are worth mentioning:

Very low drive frequency $\left(\omega\ll\omega_0\right)$

In this case, $\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\omega\Gamma\right)^2}\approx\omega_0^2$, $\tan\delta\approx 0^+\Leftrightarrow\delta\approx 0$ so $x(t)\approx\dfrac{F}{\omega_0^2}\cos\omega t.$

Very high drive frequency $\left(\omega\gg\omega_0\right)$

In this case, $\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\omega\Gamma\right)^2}\approx\omega^2$, $\tan\delta\approx 0^-\Leftrightarrow\delta\approx\pi$ so

$x(t)\approx\dfrac{F}{\omega^2}\cos(\omega t-\pi)=-\dfrac{F}{\omega^2}\cos\omega t\to 0$ as $\omega\to\infty$.

Drive frequency equal to nominal resonance frequency $\left(\omega=\omega_0\right)$

In this case, $\sqrt{\left(\omega_0^2-\omega^2\right)^2+\left(2\omega\Gamma\right)^2}=2\omega_0\Gamma$, $\tan\delta=\infty\Leftrightarrow\delta=\dfrac{\pi}{2}$ so

$x(t)=\dfrac{F}{2\omega_0\Gamma}\cos(\omega t-\tfrac{\pi}{2})=\dfrac{F}{2\omega_0\Gamma}\sin\omega t.$

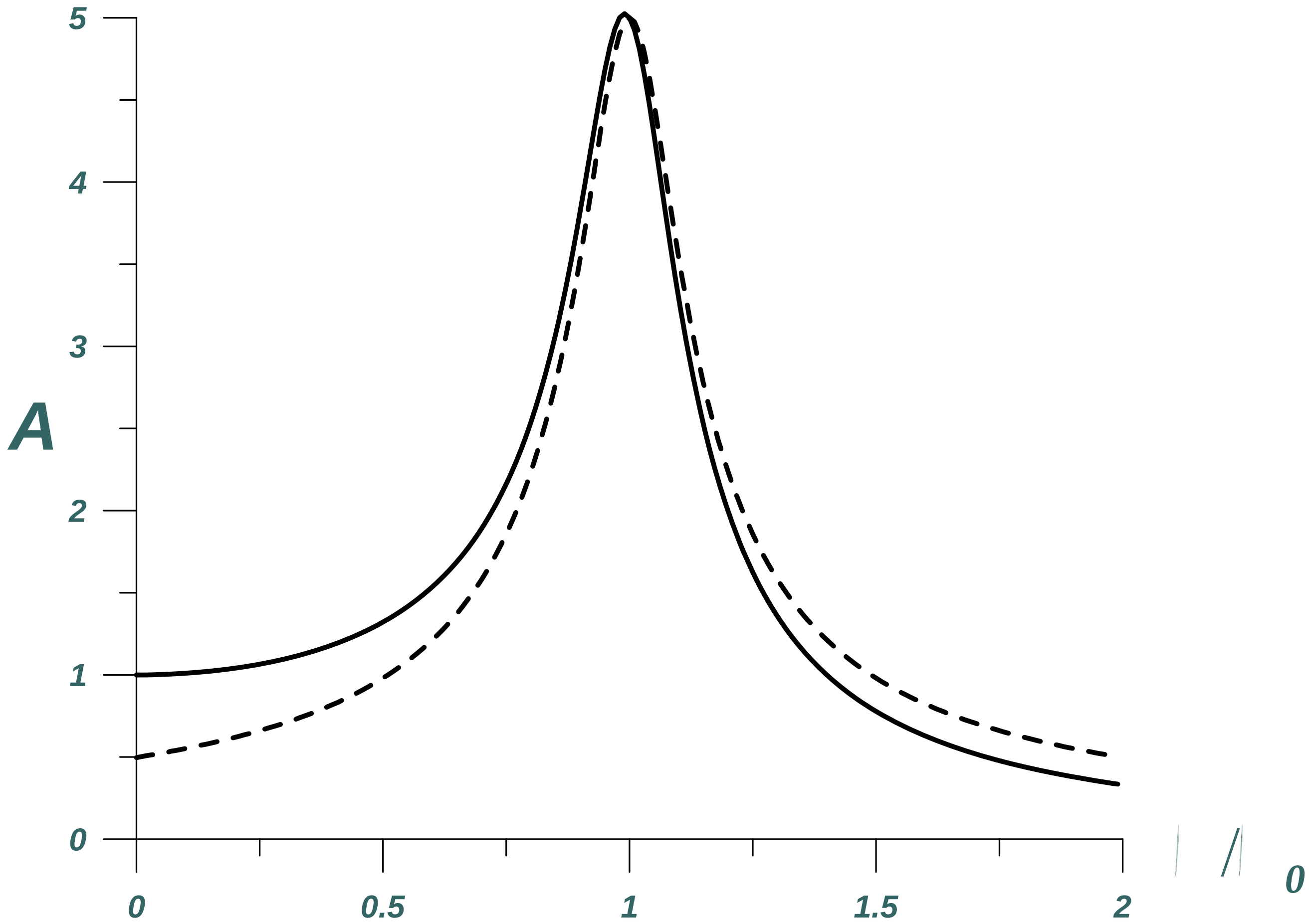
There is an important approximation to $\omega_0^2 - \omega^2 + 2i\omega\Gamma$ that applies near resonance and simplifies the mathematics considerably. When $\omega \approx \omega_0, \omega \neq \omega_0$, one can write

$$\omega_0^2 - \omega^2 + 2i\omega\Gamma = (\omega_0 + \omega)(\omega_0 - \omega) + 2i\omega\Gamma \approx (2\omega_0)(\omega_0 - \omega) + 2i\omega_0\Gamma = 2\omega_0\left[(\omega_0 - \omega) + i\Gamma\right]$$

in which case $\omega_0^2 - \omega^2 + 2i\omega\Gamma \approx 2\omega_0\sqrt{(\omega_0 - \omega)^2 + \Gamma^2}\, e^{i\delta}$, where $\tan\delta = \dfrac{\Gamma}{\omega_0 - \omega}$. Hence, the solution is

$$x(t) = \mathrm{Re}\left(\tilde{x}(t)\right) = \frac{F}{2\omega_0\sqrt{(\omega_0 - \omega)^2 + \Gamma^2}}\cos\left(\omega t - \delta\right)$$

Graphs of $\dfrac{F}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(2\omega\Gamma\right)^2}}$ and $\dfrac{F}{2\omega_0\sqrt{(\omega_0 - \omega)^2 + \Gamma^2}} = \dfrac{F}{2\omega_0^2\sqrt{(1 - \omega/\omega_0)^2 + \left(\Gamma/\omega_0\right)^2}}$ versus ω/ω_0, with $\omega_0 = 1$, F = 1, and $\Gamma/\omega_0 = 0.1$ are shown below with solid line and dotted line, respectively. The approximation gets better as Γ/ω_0 gets smaller.



PTO

Supplement – A driven 1st-order ODE: Solved with and without complex numbers

Consider the 1st-order ordinary-differential equation (ODE)

$$\frac{dx}{dt} + ax = B\cos\omega t \tag{1}$$

which is a Type C equation according to the classification of Fact Sheet 1. Notice the simple harmonic driving term with amplitude *B* on the right hand side. The parameter ω is the *angular frequency*, which is related to the natural frequency *f* and period T by $\omega = 2\pi f = 2\pi / T$ (often, the symbol ν is used instead of *f*). Please refer to question 8, Problem Sheet 1 for an earlier encounter with this equation. ☺

Solution without using complex numbers

Substituting the trial solution $x(t) = A\cos(\omega t - \phi)$, where ϕ represents a phase lag, into Eq.(1) yields

$$\begin{aligned}
&-\omega A\sin(\omega t - \phi) + aA\cos(\omega t - \phi) = B\cos\omega t \\
&\Updownarrow \\
&-\omega A\left(\sin\omega t\cos\phi - \cos\omega t\sin\phi\right) + aA\left(\cos\omega t\cos\phi + \sin\omega t\sin\phi\right) = B\cos\omega t \\
&\Updownarrow \\
&A\left(a\cos\phi + \omega\sin\phi\right)\cos\omega t + A\underbrace{\left(a\sin\phi - \omega\cos\phi\right)}_{=0}\sin\omega t = B\cos\omega t
\end{aligned} \tag{2}$$

Notice that the coefficient of $\sin\omega t$ must be set to zero for the equation to be fulfilled. Hence, it follows immediately that

$$\tan\phi = \omega / a\,. \tag{3}$$

From the coefficient of the $\cos\omega t$ terms, one now obtains

$$A = \frac{B}{a\cos\phi + \omega\sin\phi} = \frac{1}{\cos\phi}\frac{B}{\left(a + \omega\tan\phi\right)} = \sqrt{1 + \omega^2 / a^2}\,\frac{B}{\left(a + \omega^2 / a\right)} = \frac{B}{\sqrt{a^2 + \omega^2}}\,, \tag{4}$$

where we have used Eq.(3) $\sec\phi \equiv \left(\cos\phi\right)^{-1} = \sqrt{1 + \tan^2\phi} = \sqrt{1 + \omega^2 / a^2}$. Hence, we obtain

$$x(t) = A\cos(\omega t - \phi) = \frac{B}{\sqrt{a^2 + \omega^2}}\cos(\omega t - \phi)\,,\ \text{where}\ \tan\phi = \omega / a\,. \tag{5}$$

Solution using complex numbers

Introduce the complex number $\tilde{x}(t)$, where $x(t) = \mathrm{Re}\left(\tilde{x}(t)\right)$, and consider the complex equation associated with Eq.(1):

$$\frac{d\tilde{x}}{dt} + a\tilde{x} = Be^{i\omega t} . \quad (6)$$

Notice that the real part of Eq. (6) is exactly Eq.(1). Substituting the trial solution $\tilde{x} = \tilde{A}e^{i\omega t}$ into Eq.(6), we find

$$i\omega\tilde{x} + a\tilde{x} = Be^{i\omega t} \Leftrightarrow i\omega\tilde{A}e^{i\omega t} + a\tilde{A}e^{i\omega t} = Be^{i\omega t} , \quad (7)$$

so by dividing the equation through by the common (non-zero) factor $e^{i\omega t}$, we obtain

$$i\omega\tilde{A} + a\tilde{A} = B \Leftrightarrow \tilde{A} = \frac{B}{a + i\omega} . \quad (8)$$

It follows immediately that the (complex) solution to Eq. (6) is

$$\tilde{x}(t) = \tilde{A}e^{i\omega t} = \frac{B}{a + i\omega}e^{i\omega t} .$$
(9)

Since $a + i\omega = \sqrt{a^2 + \omega^2}e^{i\phi}$ where $\tan\phi = \omega / a$, Eq.(9) becomes

$$\tilde{x}(t) = \frac{B}{\sqrt{a^2 + \omega^2}}e^{i(\omega t - \phi)} \quad (10)$$

Taking the real part of $\tilde{x}(t)$, one obtains the (real) solution to the original Eq.(1)

$$x(t) = \mathrm{Re}\left(\tilde{x}(t)\right) = \frac{B}{\sqrt{a^2 + \omega^2}}\cos(\omega t - \phi) , \text{ with } \tan\phi = \omega / a \quad (11)$$

which is identical (of course) to Eq.(5).

Conclusion

It is fare to say that Eq.(1) is solved more easily using complex numbers since there is no need to remember any trigonometric relationships. Indeed, the advantage can be much greater in more complicated situations!

A Note on Notation

The symbol *B* has been used for the amplitude of the driving term on the present Fact Sheet for consistency with the lectures and question (f) of Classwork 7. However, the symbol *A* was used for this parameter in question 8 of Problem Sheet 1. This inconsistency is unfortunate and I am sure your keen eyes and brains have spotted others as well. If you have, please do let me know by sending me an e-mail k.christensen@imperial.ac.uk and I will correct the matter so future generation of Imperial College London students can benefit from your work ☺.