Fact Sheet 12

Complex numbers II: Powers, Roots & Functions

A complex number can be written in the various forms

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where $r = \sqrt{x^2 + y^2}$ is the *modulus* (or magnitude) and θ is the *argument* (or phase). Since changing the argument by multiples of 2π does not change *z*, it follows that θ should be written

$$\theta = \theta' + 2\pi n, \quad n \in \mathbb{Z}$$

where the *principal value* θ' is defined in the range $-\pi < \theta' \le \pi$ and $n \in \mathbb{Z}$ is any integer (positive, negative or zero). Clearly, $n = 0$ corresponds to the argument θ taking the principal value.

The multi-valued nature of the argument is of critical importance in what follows.

The integer power *k* of complex numbers is *single-valued*:

$$z^k = \left(re^{i\theta}\right)^k = r^k e^{ik\theta} = r^k e^{ik\theta'} e^{i2\pi nk} = r^k e^{ik\theta'} \quad \text{for integer k}, n.$$

The result does not depend on n, so z^k has only one value. Note that for $r = 1$, $\left(e^{i\theta}\right)^k = e^{ik\theta}$ yields

$$\left(\cos\theta + i\sin\theta\right)^k = \cos k\theta + i\sin k\theta$$

which is **De Moivre's Theorem**.

The rational power p/q of a complex number is *multi-valued*:

$$z^{p/q} = \left(re^{i\theta}\right)^{p/q} = r^{p/q} e^{ip\theta/q} = r^{p/q} e^{ip\theta'/q} e^{i2\pi np/q} \quad \text{for integer } p, q, n.$$

The result depends on n, $n = 0, 1, \ldots, q-1$, so $z^{p/q}$ has *q* different values, spaced apart by $e^{i2\pi p/q}$.

The q'th Root of a complex number follow from the last equation when $p = 1$. Thus

$$\sqrt[q]{z} = z^{1/q} = \left(re^{i(\theta' + 2\pi n)}\right)^{1/q} = r^{1/q} e^{i\theta'/q} e^{i2\pi n/q} \quad \text{for integer } q, n$$

The result depends on n, $n = 0, 1, \ldots, q-1$, so $\sqrt[q]{z}$ has *q* different values, spaced apart by $e^{i2\pi/q}$.

The Irrational powers of a complex number normally have infinite numbers of values.

The natural logarithm of a complex number is *multi-valued:*

$$\ln z = \ln\left(re^{i\theta}\right) = \ln r + i\theta = \ln r + i(\theta' + 2\pi n) = \text{Ln } z + i2\pi n \text{ for integer } n$$

where $\text{Ln } z \equiv \ln r + i\theta'$ is *the principal value* of $\ln z$, related to the principal value of the argument.

The complex power of a complex number follow from

$$z^c \equiv e^{c\ln z} = e^{c\left(\ln r + i\left(\theta' + 2\pi n\right)\right)} = r^c e^{ic(\theta' + 2\pi n)} = \left(r^c e^{ic\theta'}\right) e^{i2\pi nc} \text{ for complex } c.$$

The principal value of z^c is bracketed in the above expression. By convention, if *z* is real and positive, $z^c = e^{c\,\text{Ln } z}$, so only the principal value applies.

P.T.O

<u>Hyperbolic functions</u>

Taking sines and cosines of pure imaginary numbers leads to the hyperbolic functions:

Recall that

$$e^z \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} \cdots$$

$$\cos z \equiv \sum_{n=0}^{\infty} \left(-1\right)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,$$

$$\sin z \equiv \sum_{n=0}^{\infty} \left(-1\right)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^2}{3!} + \frac{5^4}{5!} - \cdots.$$

For $z = x + iy$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i}.$$

When z is *pure imaginary* (x = 0), these expressions become

$$\cos\ iy = \left(\frac{e^{y} + e^{-y}}{2}\right) = \cosh y,$$

$$\sin\ iy = i\left(\frac{e^{y} - e^{-y}}{2}\right) = i \sinh y,$$

where the terms in brackets define the *<u>hyperbolic functions</u>* cosh y and sinh y, normally pronounced “cosh” and “shine”.

Other hyperbolic functions include

Hyperbolic tan: $\tanh y = \dfrac{\sinh y}{\cosh y}$,

Hyperbolic sech: $\operatorname{sech} y = \dfrac{1}{\cosh y}$.