Fact Sheet 10 – Geometry II in 3-dimensional space

A. The equation of the line of intersection of two planes

1. Find *any* point $\mathbf{r}_0$ on the line of intersection. Do this by choosing an arbitrary value for one of the three components (say $x = 0$), and solving the two plane equations for the other two variables (y and z in this case).
2. The direction vector $\mathbf{d} \perp \mathbf{n}_1$ & $\mathbf{d} \perp \mathbf{n}_2$. Hence, the direction of the line of intersection is the cross product of normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ for the two planes, i.e., $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$.
3. The equation is $\mathbf{r} = \mathbf{r}_0 + \lambda \mathbf{d}$ in vector form. Put it in component form if you like.

B. The (acute) angle between two planes

The angle $\theta = \cos^{-1}(\hat{\mathbf{n}}_1 \bullet \hat{\mathbf{n}}_2)$, where $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are unit normal vectors to the planes.

C. The minimum distance *d* from a point *P* to a plane

1. Find *any* point A on the plane by choosing arbitrary values of two of the components and using the equation of the plane to find the third.
2. The minimum distance is $d = \left|\overrightarrow{AP} \bullet \hat{\mathbf{n}}\right|$ where $\hat{\mathbf{n}}$ is a unit normal vector.

D. The minimum distance *d* from a point *P* to a line

1. Find *any* point A on the line by choosing an arbitrary value of one the components and using the equation of the line to find the other two.
2. The minimum distance from P to the line is $d = \left|\overrightarrow{AP} \times \hat{\mathbf{d}}\right|$ where $\hat{\mathbf{d}}$ is a unit vector in the direction of the line.

E. The minimum distance *d* between two skew lines

1. Find arbitrary points A_1 and A_2 on each line, and hence the vector $\overrightarrow{A_1A_2}$ joining them.
2. Find a unit vector normal to both lines from $\hat{\mathbf{n}} = \dfrac{\mathbf{d}_1 \times \mathbf{d}_2}{\left|\mathbf{d}_1 \times \mathbf{d}_2\right|}$ where $\mathbf{d}_1$ and $\mathbf{d}_2$ are the respective direction vectors.
3. The minimum distance is $d = \left|\overrightarrow{A_1A_2} \bullet \hat{\mathbf{n}}\right|$.

F. The condition for two lines to intersect

If the minimum distance between them is zero, i.e. $d = \left|\overrightarrow{A_1A_2} \bullet \hat{\mathbf{n}}\right| = 0$, the lines intersect.

Reference Information (see Fact Sheet 4 for further details)

For the plane $ax + by + cz = k$, the two unit normal vectors are $\pm\hat{\mathbf{n}} = \pm\dfrac{\mathbf{n}}{|\mathbf{n}|} = \pm\dfrac{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}}$ and the minimum (shortest, perpendicular) distance from the origin to the plane is given by

$$d = \frac{k}{\sqrt{a^2 + b^2 + c^2}} .$$

This result can also be obtained by using recipe C in the case where P is the origin. An arbitrary point on the plane is $(k/a, 0, 0)$, which is defined by the position vector $\mathbf{r} = k\mathbf{i}/a$.

From the recipe, the minimum distance to the plane is $d = |\mathbf{r} \bullet \hat{\mathbf{n}}| = \dfrac{k}{\sqrt{a^2 + b^2 + c^2}}$ as before.