Fact Sheet 1 – Ordinary Differential Equations I

1. Terminology

- ODE stands for **O**rdinary **D**ifferential **E**quation.
- PDE stands for **P**artial **D**ifferential **E**quation.
- A PDE contains partial derivatives; an ODE does not.
- The *order* of a differential equation is the order of the highest derivative. Hence a "first-order differential equation" contains no second (or higher-order) derivatives.
- A simple first-order ODE for *x* as a function of *t* might be $\frac{dx}{dt} + ax = f(t)$. In this equation, *t* is called the *independent variable* and *x* the *dependent variable*.

2. First-Order ODEs

The general form of a first order ODE is

$$\boxed{\frac{dx}{dt} = F(x,t)} \qquad (1)$$

where $F(x,t)$ is a given function of *x* and *t*. However, an analytical solution is possible only when $F(x,t)$ has a simple structure.

Three common cases arise frequently in physical applications:

A. $$\boxed{\frac{dx}{dt} = f(t)} \qquad (2A)$$

In this case, *F* is a function of *t* only: *F(x,t)* = *f*(*t*).

B. $$\boxed{\frac{dx}{dt} = b - ax \text{ or, equivalently, } \frac{dx}{dt} + ax = b} \qquad (2B)$$

Here, *F* is a function of *x* only: *F(x,t)* = *b* - *ax*, where *a* and *b* are constants. Setting *b* = 0 gives an important special case.

C. $$\boxed{\frac{dx}{dt} + ax = f(t)} \qquad (2C)$$

In this case, $F(x,t) = f(t) - ax$ where *a* is a constant; *f(t)* might, for example, be a sine or cosine function.

In **case A**, the solution is

$$\boxed{x = \int f(t)dt} \qquad (3A)$$

which is simple enough, although it isn't always possible to perform the integration analytically. Indeed, every time you evaluate an integral, you are in effect solving a differential equation of type A! When you perform the integration, an integration constant will appear, which you must adjust to match the *boundary conditions* (see Section 3 below).

In **case B**, the solution is

$$\boxed{x = \frac{b}{a} + ce^{-at}} \qquad \boxed{\Rightarrow x = ce^{-at} \quad \text{when } b = 0.} \tag{3B}$$

where, as always, the *integration constant* c must be adjusted to fit the boundary conditions. Rather than trying to remember the solution Eq.(3B) to the ODE in case B, you should understand how to arrive to this conclusion. There are two ways this solution can be found:

(a) Separating the variables and writing the ODE in the form $\frac{dx}{b-ax} = dt$ and integrating;

(b) Multiplying the ODE through by the *integrating factor* e^{at}, which enables the equation to be written in the form $\frac{d(xe^{at})}{dt} = be^{at}$. (Convince yourself that this is the case by performing the differentiation on the left-hand-side.) It follows that $xe^{at} = \int be^{at}dt = \frac{b}{a}e^{at} + c$ from which Eq.(3B) follows immediately.

Case C is solved by using the integrating factor e^{at} as in case B. One obtains $\frac{d(xe^{at})}{dt} = f(t)e^{at}$ which leads to

$$\boxed{xe^{at} = \int f(t)e^{at}dt\,.} \tag{3C}$$

Be careful here! You can't, of course, cancel the e^{at} on the left-hand side of Eq.(3C) with the one under the integral! (Why not?)

3. *Applying Boundary Conditions*

Let's work through the solution to case B in a situation where we know that $x = x_0$ at $t = t_0$.

EITHER write $x_0 = \frac{b}{a} + ce^{-at_0}$ from Eq.(3B), which fixes the integration constant as $c = x_0 e^{at_0} - \frac{b}{a}e^{at_0}$. Plugging this back into Eq.(3B) gives

$$x = x_0 e^{-a(t-t_0)} + \frac{b}{a}(1 - e^{-a(t-t_0)})\,. \tag{4}$$

In the special case where $t_0 = 0$, Eq.(4) simplifies to

$$x = x_0 e^{-at} + \frac{b}{a}(1 - e^{-at})\,. \tag{5}$$

OR (entirely equivalently) use definite integrals and write $\left[xe^{at}\right]_{t_0}^{t} = \int_{t_0}^{t} be^{at}dt$ which leads to $xe^{at} - x_0 e^{at_0} = \frac{b}{a}\left(e^{at} - e^{at_0}\right)$. Dividing through by e^{at} gives the same result as before, as of course it must! This second method avoids any mention of c.

Applying boundary conditions to cases A and C is analogous.

Supplement: Additional Information on ODEs

This supplementary sheet contains some additional material on ODEs for future reference, as well as an important example introduced in the lectures.

A. <u>Soluble 1st order ODEs</u>

There are a small number of specific forms for $F(x,t)$ in Eq.(1) that make the 1st order ODE of Eq.(1) amenable to analytic solution. The special cases discussed above are all examples of the *<u>linear form</u>* in which $F(x,t) = h(t) - f(t)x$. The ODE then reads $\frac{dx}{dt} + h(t)x = f(t)$. Note that this type of equation can be solved by using the integrating factor $e^{\int h(t)dt}$.

Cases A-C above all involve simple versions of the linear form. In case A, $h(t) = 0$; in case C, $h(t) = b$ is a constant; and in case B, $f(t) = a$ is a constant as well. On the other hand, if $F(x,t) = f(t)g(x)$, the variables (x and t) are said to be *<u>separable</u>*. In this case, one can write $\frac{dx}{g(x)} = f(t)dt$ which can then be integrated, at least in principle. For more detailed information on first-order ODEs, see G. Stephenson, "*<u>Mathematical Methods for Science Students</u>*" (Longmans; 2nd ed. 1973) chapter 21.

B <u>Linear differential equations</u>

A differential equation is said to be *<u>linear</u>* if it contains only first powers of the dependent variable and its derivatives i.e. x, $\frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots$ but no functions, second (or higher) powers, or products. Hence, the presence of a function such as sin x, a square such as x^2, or a product such as $x\frac{dx}{dt}$ would mark the differential equation as *<u>nonlinear</u>*.