Classwork 8 – A Christmas Medley: Answers

1. The distance of closest approach is $d = \left|\overrightarrow{A_1A_2}\bullet\hat{\mathbf{n}}\right|$ where $\overrightarrow{A_1A_2}$ is <u>any</u> vector between the two paths, and $\hat{\mathbf{n}}$ is the unit normal to both of them, given by $\hat{\mathbf{n}} = \dfrac{\mathbf{d_1}\times\mathbf{d_2}}{|\mathbf{d_1}\times\mathbf{d_2}|}$, where $\mathbf{d}_1$ and $\mathbf{d}_2$ are direction vectors for the two paths.

   For Santa's path where $\mathbf{d}_1 = -\mathbf{i}-\mathbf{j}-\mathbf{k}$ we choose $A_1 = (0,0,0)$ and for the path of the wicked where $\mathbf{d}_2 = 2\mathbf{i}+5\mathbf{j}+3\mathbf{k}$ we choose $A_2 = (0,-7.5,-7.5)$ implying that $\overrightarrow{A_1A_2} = (0,-7.5,-7.5)$. Now

$$\mathbf{d}_1\times\mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \\ 2 & 5 & 3 \end{vmatrix} = (-3+5)\mathbf{i} - (-3+2)\mathbf{j} + (-5+2)\mathbf{k} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k} \text{ and } |\mathbf{d}_1\times\mathbf{d}_2| = \sqrt{14},$$

   so a unit normal vector is $\hat{\mathbf{n}} = \dfrac{2\mathbf{i}+\mathbf{j}-3\mathbf{k}}{\sqrt{14}}$.

   Hence, from the dot product, one obtains $d = \dfrac{1}{\sqrt{14}}\left|(0,-7.5,-7.5)\bullet(2,1-3)\right| = \dfrac{15}{\sqrt{14}} \approx 4$, a very comfortable margin if the units are km. Your presents seem safe (this year)!

2. Each column in the matrix **B** contains the information on how much of the raw material (flour, margarine, sugar, currants, and egg) are used for cc, lc, gb and pb, respectively:

$$\mathbf{B} = \begin{pmatrix} 0.22 & 0.22 & 0.20 & 0.22 \\ 0.15 & 0.20 & 0.08 & 0.05 \\ 0.15 & 0.18 & 0.04 & 0 \\ 0.31 & 0 & 0.12 & 0 \\ 2 & 3 & 0 & 0 \end{pmatrix}; \quad \mathbf{r} = \mathbf{Bp} \Leftrightarrow \mathbf{r} = \begin{pmatrix} 0.22 & 0.22 & 0.20 & 0.22 \\ 0.15 & 0.20 & 0.08 & 0.05 \\ 0.15 & 0.18 & 0.04 & 0 \\ 0.31 & 0 & 0.12 & 0 \\ 2 & 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 100 \\ 120 \\ 80 \\ 50 \end{pmatrix} = \begin{pmatrix} 75.4 \\ 47.9 \\ 39.8 \\ 40.6 \\ 560 \end{pmatrix}.$$

3. The matrix product $\mathbf{WW} = \begin{pmatrix} i & 1+i \\ z & -i \end{pmatrix}\begin{pmatrix} i & 1+i \\ z & -i \end{pmatrix} = \begin{pmatrix} z+zi-1 & 0 \\ 0 & z+zi-1 \end{pmatrix}$.

If $z = x+iy$, then $z+zi-1 = x+iy+ix-y-1 = (x-y-1)+i(x+y)$ and the matrix product

$$\mathbf{WW} = \begin{pmatrix} (x-y-1)+i(x+y) & 0 \\ 0 & (x-y-1)+i(x+y) \end{pmatrix}.$$

(a) Both the real and the imaginary part of the entries in the main diagonal are zero, yielding $x = \frac{1}{2}, y = -\frac{1}{2}$, that is, $z = \frac{1}{2} - i\frac{1}{2} = \frac{1}{2}(1-i)$.

(b) Here, $x-y-1=1$ and $x+y=0$, implying $x=1, y=-1$, that is, $z = 1-i$.

(c) Inserting $z=1-i$ into the matrix, we have $\mathbf{W}=\begin{pmatrix} i & 1+i \\ 1-i & -i \end{pmatrix}$ so $\mathbf{W}^*=\begin{pmatrix} -i & 1-i \\ 1+i & i \end{pmatrix}$ and multiplying the two matrices together yields

$$\mathbf{WW}^*=\begin{pmatrix} i & 1+i \\ 1-i & -i \end{pmatrix}\begin{pmatrix} -i & 1-i \\ 1+i & i \end{pmatrix}=\begin{pmatrix} -ii+(1+i)^2 & i(1-i)+i(1+i) \\ -i(1-i)-i(1+i) & (1-i)^2-ii \end{pmatrix}=\begin{pmatrix} 1+2i & 2i \\ -2i & 1-2i \end{pmatrix}.$$

4. To find the eigenvalues, we solve the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$ and to find the associated eigenvectors, we solve the homogeneous equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ for each λ.

(i) The characteristic eq. $\det(\mathbf{A}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix}=0 \Leftrightarrow (1-\lambda)(2-\lambda)-2\cdot 3=0$, that is $\lambda^2-3\lambda-4=0 \Leftrightarrow \lambda=\dfrac{3\pm\sqrt{3^2-4\cdot 1\cdot(-4)}}{2}=\begin{cases} 4 \\ -1 \end{cases}$ so the eigenvalues are $\lambda_1=4, \lambda_2=-1$. We find

$\lambda_1=4$: $\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3x_1+3y_1=0 \\ 2x_1-2y_1=0 \end{cases} \Leftrightarrow x_1=y_1$ yielding an eigenvector $\mathbf{x}_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\lambda_2=-1$: $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2x_2+3y_2=0 \\ 2x_2+3y_2=0 \end{cases} \Leftrightarrow y_2=-\tfrac{2}{3}x_2$ yielding an eigenvector $\mathbf{x}_2=\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

(ii) We solve $\det(\mathbf{B}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}=0 \Leftrightarrow (1-\lambda)\left((2-\lambda)^2-9\right)=0$, and thus $\lambda^3-5\lambda^2-\lambda+5=(\lambda-5)(\lambda^2-1)=0$, so the eigenvalues are $\lambda_1=5, \lambda_2=1, \lambda_3=-1$. We find

$\lambda_1=5$: $\begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3x_1+3y_1=0 \\ 3x_1-3y_1=0 \\ -4z_1=0 \end{cases} \Leftrightarrow \begin{cases} x_1=y_1 \\ z_1=0 \end{cases}$ so $\mathbf{x}_1=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

$\lambda_2=1$: $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_2+3y_2=0 \\ 3x_2-3y_2=0 \\ 0=0 \end{cases} \Leftrightarrow \{x_2=y_2=0$ so $\mathbf{x}_2=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector.

$\lambda_3=-1$: $\begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 3x_3+3y_3=0 \\ 3x_3+3y_3=0 \\ 2z_3=0 \end{cases} \Leftrightarrow \begin{cases} y_3=-x_3 \\ z_3=0 \end{cases}$ so $\mathbf{x}_3=\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector.

(iii) The characteristic equation $\det(\mathbf{C}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}=0$. Evaluating the determinant, we find $\lambda^3-4\lambda^2-\lambda+4=(\lambda-4)(\lambda-1)(\lambda+1)=0$, so the three eigenvalues are $\lambda_1=4, \lambda_2=1, \lambda_3=-1$. To find the associated eigenvectors we solve the homogeneous equations:

$$\lambda_1 = 4: \begin{pmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3x_1 + 2y_1 + z_1 = 0 \\ 2x_1 - 3y_1 + z_1 = 0 \\ x_1 + y_1 - 2z_1 = 0 \end{cases} \Leftrightarrow \left\{ x_1 = y_1 = z_1 \right. .$$

Therefore, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1 = 4.$.

$$\lambda_2 = 1: \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2y_2 + z_2 = 0 \\ 2x_2 + z_2 = 0 \\ x_2 + y_2 + z_2 = 0 \end{cases} \Leftrightarrow \left\{ x_2 = y_2 = -\tfrac{1}{2} z_2 \right. .$$

Hence, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2 = 1$.

$$\lambda_3 = -1: \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2x_3 + 2y_3 + z_3 = 0 \\ 2x_3 + 2y_3 + z_3 = 0 \\ x_3 + y_3 + 3z_3 = 0 \end{cases} \Leftrightarrow \begin{cases} y_3 = -x_3 \\ z_3 = 0 \end{cases} .$$

Therefore, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_3 = -1$.