Classwork 8 – A Christmas Medley

- Enter into the spirit of the festive season with question 1.
- Encounter a mix of cooking and matrices in question 2.
- Take an excursion into complex matrices in question 3.
- Go hunting for eigenvectors and eigenvalues in question 4.

I wish you all a Merry Christmas and a Happy New Year – may you all enjoy the winter-break!

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1. Santa's sleigh is approaching Imperial College London, which is, of course, the origin of all right-handed coordinate systems. Santa, coming from Lapland (personally, I believe Santa resides in Greenland!), is heading in a southwesterly direction and is descending steeply on the track $\mathbf{r} = \lambda(-\mathbf{i} - \mathbf{j} - \mathbf{k})$ where x is east, y is north, and z is the upwards vertical. The wicked witch/DarthVader/Tash/Voldemort/Sauron/Kim ……………………… (enter your favourite choice) is rising rapidly from the depths on the path determined by $\mathbf{r} = -7.5(\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$ to intercept Santa and steal your presents!

   Find the distance of closest approach of the two tracks. Units are à choix – you may want to make them kilometres to ensure that Santa arrives safely with your presents ☺.

   [Should you need a hint, please see Fact Sheet 10.]

2. *I am indebted to Dr Michael Coppins for this next question.*

   Boltzmann's Bakery uses the finest flour, margarine, sugar, currants, and eggs to make its celebrated range of products:

   - **Alternating Currant Cake** (*cc* made per day)
     Each cake uses 0.22 kg flour, 0.15 kg margarine, 0.15 kg sugar, 0.31 kg currants, and 2 eggs.
   - **Heaviside Layer Cake** (*lc* made per day)
     Each cake uses 0.22 kg flour, 0.20 kg margarine, 0.18 kg sugar, and 3 eggs.
   - **Fermat's Last Garibaldi Biscuits (***gb* made per day)
     Each packet uses 0.20 kg flour, 0.08 kg margarine, 0.04 kg sugar, and 0.12 kg currants.
   - **XY Plain Biscuits** (*pb* made per day)
     Each packet uses 0.22 kg flour, and 0.05 kg margarine.

   The vector $\mathbf{p} = \begin{pmatrix} cc \\ lc \\ gb \\ pb \end{pmatrix}$ contains the numbers of each product made per day while $\mathbf{r} = \begin{pmatrix} r_f \\ r_m \\ r_s \\ r_c \\ r_e \end{pmatrix}$ represents the quantity of raw materials used per day, where the indices refer to f = flour, m = margarine, s = sugar, c = currants, and e = eggs.

   Write down the matrix **B** such that $\mathbf{r} = \mathbf{Bp}$, and evaluate **r** for $\mathbf{p} = \begin{pmatrix} 100 \\ 120 \\ 80 \\ 50 \end{pmatrix}$.

3. In this question, the matrix **W** has complex elements. The *complex conjugate matrix* $\mathbf{W}^*$ is obtained by taking the complex conjugate of every element of **W**.

The matrix **W** is defined as $\mathbf{W} = \begin{pmatrix} i & 1+i \\ z & -i \end{pmatrix}$ where z is a complex number.

(a) Find z in the case where $\mathbf{WW} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) Find z in the case where $\mathbf{WW} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) In the latter case where $\mathbf{WW} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find $\mathbf{WW}^*$.

4. Find the eigenvalues and eigenvectors of the matrices

(i) $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ (ii) $\mathbf{B} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (iii) $\mathbf{C} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Un-normalised eigenvectors will do. Note that in (ii) and (iii), you will have to solve a cubic equation for the eigenvalues. However, the eigenvalues are (conveniently) integers in both cases. Therefore, once you have found one of them, λ_1, by inspection, you can divide the cubic equation through by $(\lambda - \lambda_1)$ and solve the resulting quadratic equation to find λ_2 and λ_3.