Classwork 7 – Complex Oscillations: *Answers*

(a) (i) If $\tilde{\omega} = \omega \in \mathbb{R}$, then $\tilde{x}(t) = Ae^{i\omega t} = A(\cos\omega t + \sin\omega t)$ so $x(t) = \text{Re}(\tilde{x}(t)) = A\cos\omega t$.

(ii) If $\tilde{\omega} = i\gamma,\ \gamma \in \mathbb{R}$, then $\tilde{x}(t) = Ae^{i(i\gamma)t} = Ae^{-\gamma t}$, so $x(t) = \text{Re}(\tilde{x}(t)) = Ae^{-\gamma t}$ which represents exponential decay if γ is positive and exponential growth if γ is negative.

(b) We note that $\tilde{x}(t) = \tilde{A}e^{i\omega t} = Ae^{i\theta}e^{i\omega t} = Ae^{i(\omega t+\theta)}$.

(i) For $\theta = -\frac{\pi}{2}$, we have $x(t) = \text{Re}(\tilde{x}(t)) = A\cos(\omega t - \pi/2) = A\sin\omega t$. The latter result also follows from noting that $e^{-i\pi/2} = -i$.

(ii) For $\theta = \pm\pi$, we have $x(t) = \text{Re}(\tilde{x}(t)) = A\cos(\omega t \pm \pi) = -A\cos\omega t$. The latter result also follows from noting that $e^{\pm i\pi} = -1$.

(c) If $\tilde{\omega} = \omega + i\gamma,\ \omega,\gamma \in \mathbb{R}$, then $\tilde{x}(t) = Ae^{i\tilde{\omega}t} = Ae^{i\omega t - \gamma t} = Ae^{-\gamma t}e^{i\omega t}$, so $x(t) = Ae^{-\gamma t}\cos\omega t$.

(i) $x(0) = Ae^0\cos 0 = A$.

(ii) The oscillation period T is defined by $\cos\omega t = \cos(\omega(t+T)) = \cos(\omega t + \omega T)$, and hence $\omega T = 2\pi$ yielding $T = 2\pi/\omega$.

(iii) By definition, the amplitude of the oscillation is given by $Ae^{-\gamma t}$. The initial amplitude (at $t = 0$) is A, so the equation to determine $t_{\frac{1}{2}}$ is: $Ae^{-\gamma t_{\frac{1}{2}}} = \frac{1}{2}A$. Taking the natural logarithm, we find $-\gamma t_{\frac{1}{2}} = \log_e 1 - \log_e 2 = -\log_e 2$, so $t_{\frac{1}{2}} = \frac{\log_e 2}{\gamma}$.

(d) (i) Inserting the trial solution $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$ into the complex equation $m\frac{d^2\tilde{x}}{d\tilde{x}^2} = -k\tilde{x}$ yields $m(i\omega_0)^2\tilde{x} = -k\tilde{x}$, that is, $m\omega_0{}^2 = k$ so the natural angular frequency $\omega_0 = \sqrt{k/m}$. This makes sense qualitatively speaking: The larger the spring constant, the larger the frequency and, the larger the mass, the lesser the frequency. Also note that since $\left[\frac{k}{m}\right] = \frac{kgs^{-2}}{kg} = s^{-2}$, dimensional considerations alone reveals that $\omega_0 = const \times \sqrt{k/m}$ since $[\omega_0] = s^{-1}$.

(ii) The (complex) speed $\tilde{v}(t) = \frac{d\tilde{x}(t)}{dt} = i\omega_0\tilde{x}(t) = i\omega_0\tilde{A}e^{i\omega_0 t}$.

(e) We have that $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$ (with $\omega_0 = \sqrt{k/m}$) and $\tilde{v}(t) = \frac{d\tilde{x}}{dt} = i\omega_0\tilde{A}e^{i\omega_0 t}$.

(i) If $\tilde{A} = A \in \mathbb{R}$, then the (real) displacement $x(t) = \text{Re}(\tilde{x}(t)) = \text{Re}(Ae^{i\omega_0 t}) = A\cos\omega_0 t$ and the (real) speed $v(t) = \text{Re}(\tilde{v}(t)) = \text{Re}(i\omega_0 Ae^{i\omega_0 t}) = -\omega_0 A\sin\omega_0 t$.

(ii) If $\tilde{A} = Ae^{i\pi/2} = iA$, then the displacement $x(t) = \text{Re}(\tilde{x}(t)) = \text{Re}(iAe^{i\omega_0 t}) = -A\sin\omega_0 t$ and the speed $v(t) = \text{Re}(\tilde{v}(t)) = \text{Re}(i\omega_0 iAe^{i\omega_0 t}) = -\omega_0 A\cos\omega_0 t$.

(f) (i) The associated complex differential equation is $\frac{d\tilde{x}}{dt} + a\tilde{x} = Be^{i\omega t}$.

(ii) Substituting the trial solution $\tilde{x}(t) = \tilde{A}e^{i\omega t}$ into $\frac{d\tilde{x}}{dt} + a\tilde{x} = Be^{i\omega t}$ yields $i\omega\tilde{A}e^{i\omega t} + a\tilde{A}e^{i\omega t} = Be^{i\omega t}$. Dividing by $e^{i\omega t}$ we have $i\omega\tilde{A} + a\tilde{A} = B$ with the solution $\tilde{A} = \frac{B}{a + i\omega}$. Substituting this back into the trial solution we find $\tilde{x}(t) = \frac{B}{a + i\omega}e^{i\omega t}$.

In order to facilitate taking the real part, we multiply and divide by $a - i\omega$:

$\tilde{x}(t) = \frac{B(a - i\omega)}{a^2 + \omega^2}e^{i\omega t} = \frac{Be^{-i\phi}}{\sqrt{a^2 + \omega^2}}e^{i\omega t} = \frac{B}{\sqrt{a^2 + \omega^2}}e^{i(\omega t - \phi)}$ where $e^{-i\phi} \equiv \frac{a - i\omega}{\sqrt{a^2 + \omega^2}}$ and (since $a, \omega > 0$) $\tan\phi = \omega / a$. Hence we find that the (real) displacement

$x(t) = \mathrm{Re}\left(\tilde{x}(t)\right) = \frac{B}{\sqrt{a^2 + \omega^2}}\cos(\omega t - \phi)$ where $\tan\phi = \omega / a$.

(iii) When $\omega \ll a$, then $a^2 + \omega^2 \approx a^2$ and $\tan\phi \approx 0 \Rightarrow \phi \approx 0$, so $x(t) \approx \frac{B}{a}\cos\omega t$.

(iv) When $\omega \gg a$, then $a^2 + \omega^2 \approx \omega^2$ and $\tan\phi \to \infty \Rightarrow \phi \to \frac{\pi}{2}$, so

$x(t) \approx \frac{B}{\omega}\cos\left(\omega t - \pi / 2\right) = \frac{B}{\omega}\sin\omega t$.

(v) When $\omega = a$, then $a^2 + \omega^2 = 2a^2$ and $\tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}$, so $x(t) = \frac{B}{a\sqrt{2}}\cos\left(\omega t - \pi / 4\right)$.

(vi) The amplitude is given by $\frac{B}{\sqrt{a^2 + \omega^2}}$. For fixed a, the amplitude decreases with increasing frequency ω. The amplitude of the oscillations is *B/a* at $\omega = 0$, $B / a\sqrt{2}$ at $\omega = a$, B / ω for $\omega \gg a$, and tends to zero (like $1/\omega$) as $\omega \to \infty$.

(h) (i) $\omega_0 = \sqrt{k / m} = \sqrt{100 / 4}\,\mathrm{rads}^{-1} = 5\,\mathrm{rads}^{-1}$;

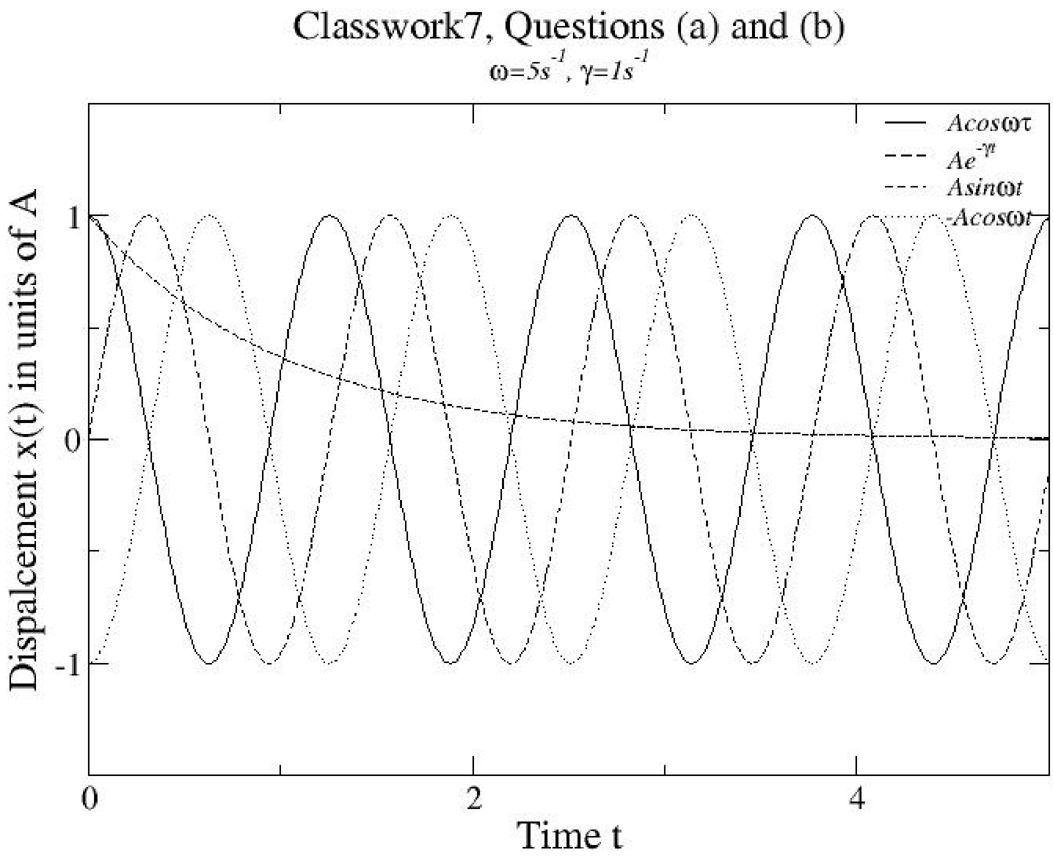
(ii) $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{5}s = 1.26\,\mathrm{s}$;

(iii) Since $A \in \mathbb{R}$, the displacement $x(t) = A\cos\omega_0 t$ and the speed $v(t) = -\omega_0 A\sin\omega_0 t$.

At t = 0, $x(0)$ = 0.1 m, v = 0 m/s.

At t = 0.2 s, x = 0.1 cos(1.0) = 0.054 m, v = –0.5 sin(1.0) = –0.421 m/s.

At t = 0.4 s, x = 0.1 cos(2.0) = –0.042 m, v = –0.5 sin(2.0) = –0.455 m/s.


Classwork7, Questions (a) and (b)
ω=5s-1, γ=1s-1
Acosωτ
Ae-γt
Asinωt
-Acosωt
1
0
-1
Dispalcement x(t) in units of A
0
2
4
Time t