Mathematical Techniques II: Algebra (Physics Year 1 Term 1) **CW7 07/12/2007**

Classwork 7 – Complex Oscillations

Background

Problems involving oscillations, especially those involving the associated differential equations, are most elegantly solved using complex numbers. In these applications, the physical quantities and parameters are themselves real, but using complex numbers makes the mathematics easier.

For example, consider a mass on a spring, which is displaced from its equilibrium position and allowed to oscillate. We are interested in the displacement *x(t)* of the mass from equilibrium as a function of time. Clearly, $x(t)$ is real, but it is convenient to treat it as the real part of a complex displacement $\tilde{x}(t)$, because this makes it easier to solve the associated equations. Once a solution $\tilde{x}(t)$ has been obtained, one takes the real part of that solution to get back to $x(t)$, that is, $x(t) = \mathrm{Re}(\tilde{x}(t))$.

Complex numbers make these problems easier because $e^{i\omega t}$ is simpler to handle than $\cos\omega t$ and $\sin\omega t$. Why? (A) Apart from a factor, it differentiates into itself; for example $\frac{d(e^{i\omega t})}{dt} = i\omega e^{i\omega t}$ and $\frac{d^2(e^{i\omega t})}{dt^2} = (i\omega)^2 e^{i\omega t} = -\omega^2 e^{i\omega t}$. (B) With $\omega = i\gamma,\ \gamma \in \mathbb{R}$, it describes naturally exponential decay for $\gamma > 0$ and exponential growth for $\gamma < 0$. (C) A phase change is readily introduced by multiplying $e^{i\omega t}$ by $e^{i\theta}$ to yield $e^{i(\omega t+\theta)}$. (D) $e^{i\omega t} \neq 0$ and therefore you can always divided an equation through by $e^{i\omega t}$ to simplify it.

This classwork introduces you to the use of complex numbers for representing oscillations. In the following, $x(t), A, \omega \in \mathbb{R}$ are real quantities (displacement, amplitude and angular frequency) while the associated complex quantities are identified by tildes, $\tilde{x}(t), \tilde{A}, \tilde{\omega} \in \mathbb{C}$.

(a) The position $x(t)$ of an object as a function of time is given by the real part of $\tilde{x}(t) = Ae^{i\tilde{\omega}t}$, where $A \in \mathbb{R}$ and $\tilde{\omega}$ are constants. Sketch the motion if

(i) $\tilde{\omega}$ is real, that is, $\tilde{\omega} = \omega$ where $\omega \in \mathbb{R}$,

(ii) $\tilde{\omega}$ is pure imaginary, that is, $\tilde{\omega} = i\omega$ where $\omega \in \mathbb{R}$.

(b) If $\tilde{x}(t) = \tilde{A}e^{i\omega t}$ where $\tilde{A} = Ae^{i\theta}$, sketch the motion when

(i) $\theta = -\pi/2$

(ii) $\theta = \pm\pi$.

(c) If $\tilde{\omega}$ is complex, that is, $\tilde{\omega} = \omega + i\gamma$ where $\omega, \gamma \in \mathbb{R}$, the motion has the form of damped oscillations in which the amplitude of the oscillations decays exponentially with time. Write down expressions for

(i) $x(t)$ at $t = 0$,

(ii) the oscillation period T

(iii) the time $t_{\frac{1}{2}}$ taken for the amplitude of the oscillations to decay to half its value.

(d) Ignoring friction, Newton's second law of motion for an object of mass m attached to a spring of spring constant k is $m\frac{d^2x}{dt^2} = -kx$, where x is the displacement from equilibrium; The complex analogue of the equation is $m\frac{d^2\tilde{x}}{dt^2} = -k\tilde{x}$.

Let $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$ denote a so-called *trial solution*.

(i) Substitute $\tilde{x}(t)$ into the complex differential equation to find the natural angular frequency ω_0 in terms of m and k.

(ii) Write down an expression for the speed $\tilde{v}(t)$ of the object in complex form.

(e) (i) In the case where $\tilde{A} = A$ (real), take the real part of the complex variables to show that the physical displacement and speed at time t are $x(t) = A\cos\omega_0 t$ and $v(t) = -A\omega_0 \sin\omega_0 t$.

(ii) Repeat for the case where $\tilde{A} = Ae^{i\pi/2}$.

(f) Consider the driven first-order ODE from Problem Sheet 1, question 8, namely

$$\frac{dx}{dt} + ax = B\cos\omega t\,.$$

This equation is directly applicable to several real problems in physics, for example, in electronics when an AC voltage of amplitude V and angular frequency ω is applied to a resistor R and an inductor L in series. In that case, $x = I$ (current), $a = R/L$, and $B = V/L$.

(i) Write down the corresponding complex equation.

(ii) By introducing the trial solution $\tilde{x}(t) = \tilde{A}e^{i\omega t}$, show that $\tilde{A} = \frac{B}{a + i\omega}$ and hence that

$$x(t) = \frac{B}{\sqrt{a^2 + \omega^2}}\cos(\omega t - \phi) \text{ where } \tan\phi = \omega / a\,.$$

(iii) Show that $x(t) \approx \frac{B}{a}\cos\omega t$ when $\omega \ll a$.

(iv) Show that $x(t) \approx \frac{B}{\omega}\sin\omega t$ when $\omega \gg a$.

(v) Find the solution when $\omega = a$.

(vi) How does the amplitude of x change as the frequency ω is increased from zero at fixed a?

(g) Return to parts (d)-(e). For m = 4kg, k =100N/m and A = 0.1m, find

(i) the angular frequency ω_0,

(ii) the period of oscillation,

(iii) the displacement and speed at t = 0s, 0.2s, and 0.4s.