Classwork 6 – Transforming Areas & Volumes: *Answers*

(a) $\mathbf{r}_2 = \mathbf{T}\mathbf{r}_1 \Leftrightarrow \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_1x_1 + b_1y_1 \\ a_2x_1 + b_2y_1 \end{pmatrix}$, that is, $x_2 = a_1x_1 + b_1y_1$; $y_2 = a_2x_1 + b_2y_1$.
The origin $x_1 = y_1 = 0$ is transformed into itself, $x_2 = y_2 = 0$.

(b) Just by inspection of the Figure, we find $\mathbf{r}_\mathrm{A} = \begin{pmatrix} u \\ v \end{pmatrix}, \mathbf{r}_\mathrm{B} = \begin{pmatrix} u+s \\ v \end{pmatrix}, \mathbf{r}_\mathrm{C} = \begin{pmatrix} u+s \\ v+s \end{pmatrix}, \mathbf{r}_\mathrm{D} = \begin{pmatrix} u \\ v+s \end{pmatrix}$.

Similarly, by inspection, $\overrightarrow{AB} = \overrightarrow{DC} = s\mathbf{i}, \overrightarrow{AD} = \overrightarrow{BC} = s\mathbf{j}.$

(c) Consider a line $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}$. Since the transformation is linear, we find that
$\mathbf{Tr} = \mathbf{T}(\mathbf{r}_0 + \lambda\mathbf{d}) = \mathbf{Tr}_0 + \mathbf{T}(\lambda\mathbf{d}) = \mathbf{Tr}_0 + \lambda(\mathbf{Td})$ which indeed is a straight line passing through the point $\mathbf{Tr}_0$ and direction vector $\mathbf{Td}$.

(d) We find that the corners of the square transform into

$$\mathbf{r}_E = \mathbf{Tr}_A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\begin{pmatrix} a_1u + b_1v \\ a_2u + b_2v \end{pmatrix}, \quad \mathbf{r}_F = \mathbf{Tr}_B = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u+s \\ v \end{pmatrix}\begin{pmatrix} a_1(u+s) + b_1v \\ a_2(u+s) + b_2v \end{pmatrix},$$

$$\mathbf{r}_G = \mathbf{Tr}_C = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u+s \\ v+s \end{pmatrix}\begin{pmatrix} a_1(u+s) + b_1(v+s) \\ a_2(u+s) + b_2(v+s) \end{pmatrix}, \text{ and}$$

$$\mathbf{r}_H = \mathbf{Tr}_D = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u \\ v+s \end{pmatrix}\begin{pmatrix} a_1u + b_1(v+s) \\ a_2u + b_2(v+s) \end{pmatrix}, \text{ respectively.}$$

Using $\overrightarrow{EF} = \mathbf{r}_F - \mathbf{r}_E, \overrightarrow{HG} = \mathbf{r}_G - \mathbf{r}_H, \overrightarrow{EH} = \mathbf{r}_H - \mathbf{r}_E, \overrightarrow{FG} = \mathbf{r}_G - \mathbf{r}_F$, , we find

$$\overrightarrow{EF} = \overrightarrow{HG} = \begin{pmatrix} a_1s \\ a_2s \end{pmatrix}, \quad \overrightarrow{EH} = \overrightarrow{FG} = \begin{pmatrix} b_1s \\ b_2s \end{pmatrix}.$$

(e) We evaluate the results above using $\mathbf{T} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}, \mathbf{r}_A = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $s = 3$, yielding

$\mathbf{r}_E = \begin{pmatrix} -5 \\ -6 \end{pmatrix}, \mathbf{r}_F = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{r}_G = \begin{pmatrix} 10 \\ 12 \end{pmatrix}, \mathbf{r}_H = \begin{pmatrix} 1 \\ 6 \end{pmatrix}, \overrightarrow{EF} = \overrightarrow{HG} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}; \overrightarrow{EH} = \overrightarrow{FG} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$. See next pg.

(f) (i) Since $\overrightarrow{EF} = a_1s\mathbf{i} + a_2s\mathbf{j}$ and $\overrightarrow{EH} = b_1s\mathbf{i} + b_2s\mathbf{j}$, the area of the parallelogram is

$$\left|\overrightarrow{EF} \times \overrightarrow{EH}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1s & a_2s & 0 \\ b_1s & b_2s & 0 \end{matrix} \right\| = |a_1sb_2s - b_1sa_2s| = s^2\,|(a_1b_2 - a_2b_1)| = s^2\,|\det\mathbf{T}|.$$

Hence, since the original area was s^2, the area scale factor is multiplied by $|\det\mathbf{T}|$. (Note that, in the equation for the area, the outer bars signify the absolute value while the inner bars signify the determinant.) (ii) Inserting the values, we find $|\det\mathbf{T}| = |12 - 4| = 8.$

(g) Yes, because any shape can be considered to be an assembly of small squares.

(h) (i) The transformations of the natural basis vectors $f(\mathbf{e}_j)$ are the column vectors $\mathbf{a}_j$ of the matrix defining the transformation:

$$\mathbf{a}_1 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{a}_2 = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \quad \mathbf{a}_3 = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

(ii) The volume is $\left|\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)\right| = \left\|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right\| = \left|\det \mathbf{T}\right|$ as before.

(iii) Yes, because any solid can be considered to be an assembly of small cubes.

(e) Sketch of square *ABCD* and its transform, the parallelogram *EFGH:*