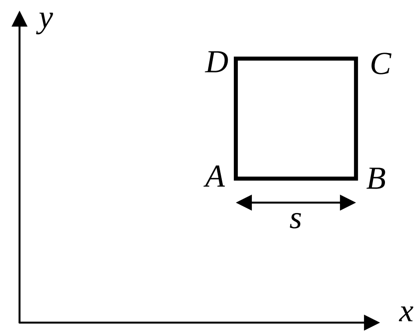
Classwork 6 – Transforming Areas and Volumes

First we consider a transformation of areas, that is, a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) The matrix $\mathbf{T} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ transforms a point defined by the position vector $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ into a new point $\mathbf{r}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ defined by $\mathbf{r}_2 = \mathbf{T}\mathbf{r}_1$. Write down equations for x_2 and y_2 in terms of x_1 and y_1, and the elements of **T**. How is the origin transformed?

(b) *ABCD* is a square of side *s* as shown in the Fig. with the lower left-hand corner *A* at position $\mathbf{r}_A = \begin{pmatrix} u \\ v \end{pmatrix}$.



Write down expressions for the vectors $\mathbf{r}_B, \mathbf{r}_C$, and $\mathbf{r}_D$ defining the other three corners *B, C,* and *D.*

Find the vectors $\overrightarrow{AB}, \overrightarrow{DC}, \overrightarrow{AD}, \overrightarrow{BC}$.

(c) The vector equation of a straight line through a point $\mathbf{r}_0$ and direction $\mathbf{d}$ is $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}$. Convince yourself that **T** transforms one straight line into another straight line.

(d) It follows from (c) that **T** transforms *ABCD* into a quadrilateral *EFGH*. Write down the position vectors $\mathbf{r}_E, \mathbf{r}_F, \mathbf{r}_G, \mathbf{r}_H$ defining all four corners of *EFGH*, and hence find the vectors $\overrightarrow{EF}, \overrightarrow{HG}, \overrightarrow{EH}, \overrightarrow{FG}$. You should find that $\overrightarrow{EF} = \overrightarrow{HG}$ and $\overrightarrow{EH} = \overrightarrow{FG}$, which implies that opposite sides of the quadrilateral are equal in length and direction, and hence that the quadrilateral is, in fact, a parallelogram.

(e) If $\mathbf{T} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$, $\mathbf{r}_A = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, and *s* = 3, find the corners of the parallelogram and the vectors $\overrightarrow{EF}, \overrightarrow{HG}, \overrightarrow{EH}, \overrightarrow{FG}$. Make a rough sketch of the square *ABCD* and its transform *EFGH*.

(f) The area of a parallelogram is $|\mathbf{A} \times \mathbf{B}|$ where **A** and **B** are the two vectors defining the two adjacent sides. The parallelogram lies in the *x*-*y* plane, that is, $A_z = B_z = 0$, and hence the area is $|A_x B_y - A_y B_x|$.

(i) Find the area of *EFGH*, and show that the area scale factor for the transformation **T**, i.e, the factor by which the area of the original square is multiplied, is $|\det \mathbf{T}| = |a_1 b_2 - a_2 b_1|$.

(ii) Put in the numbers from the previous question.

(g) Does the same area scale factor apply to other 2D shapes transformed by **T**?

P.T.O.

Now we consider a transformation of volumes, that is, a transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(h) These ideas above can be extended to 3D (indeed to any dimension). A unit cube has edges of unit length parallel to the coordinate axes and one corner is at the origin. The linear transformation $\mathbf{T} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ transforms the cube into a parallelepiped.

(i) Write down the vectors representing the three edges of the parallelepiped that intersect at the origin.

(ii) By using the formula from Fact Sheet 9 for the volume of a parallelepiped, find the volume scale factor for the transformation applied to the cube.

(iii) Does the same volume scale factor apply to other three-dimensional shapes transformed by $\mathbf{T}$?

The examples above illustrates the following general theorems:

Theorem 1: A linear function f from $\mathbb{R}^n$ to $\mathbb{R}^n$ with matrix $\mathbf{A}$ multiplies volumes by the factor $|\det \mathbf{A}|$.

Theorem 2: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then the associated matrix

$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \equiv (\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n)$ where the *j*th column $\mathbf{a}_j = f(\mathbf{e}_j),\ j = 1, 2, \ldots n$ is the transformation of the *j*th natural basis vector $\mathbf{e}_j$.

Theorem 3: Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be n vectors in $\mathbb{R}^n$. Then the volume of the parallelepiped with edges $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is $|\det(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)|$.