Classwork 5 – Discover the Orthogonal Matrix Answers

(a) (i) The magnitude of $\mathbf{u}$ is the root of the sum of the squares of the components, that is, $|\mathbf{u}| = \sqrt{u_x^2 + u_y^2} = 1 \Leftrightarrow u_x^2 + u_y^2 = 1$. Identical argument for $\mathbf{v}$ yields $u_x^2 + u_y^2 = v_x^2 + v_y^2 = 1$.

The two vectors are orthogonal if $\mathbf{u} \bullet \mathbf{v} = 0 \Leftrightarrow \begin{pmatrix} u_x \\ u_y \end{pmatrix} \bullet \begin{pmatrix} v_x \\ v_y \end{pmatrix} = u_x v_x + u_y v_y = 0$.

(ii) In matrix form, the conditions are $u_x^2 + u_y^2 = \begin{pmatrix} u_x & u_y \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \mathbf{u}^t \mathbf{u} = \mathbf{v}^t \mathbf{v} = 1$ and

$u_x v_x + u_y v_y = \begin{pmatrix} u_x & u_y \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x & v_y \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \mathbf{u}^t \mathbf{v} = \mathbf{v}^t \mathbf{u} = 0$, respectively.

(b) Since $|\mathbf{a}| = \sqrt{3^2 + (-4)^2} = 5$, $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$, so $\hat{\mathbf{b}}_1 = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$ and $\hat{\mathbf{b}}_2 = -\hat{\mathbf{b}}_1 = \begin{pmatrix} -4/5 \\ -3/5 \end{pmatrix}$.

We easily check that $\hat{\mathbf{a}} \bullet \hat{\mathbf{b}}_1 = \hat{\mathbf{a}} \bullet \hat{\mathbf{b}}_2 = 0$ and (clearly) $|\hat{\mathbf{b}}_i|^2 = \hat{\mathbf{b}}_i \bullet \hat{\mathbf{b}}_i = 1, i = 1, 2$.

(c) We find that $\mathbf{O}^t \mathbf{O} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} u_x^2 + u_y^2 & u_x v_x + u_y v_y \\ v_x u_x + v_y u_y & v_x^2 + v_y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ using the conditions in (a).

(d) (i) We find $\mathbf{q} = \mathbf{A}\mathbf{p} \Leftrightarrow \begin{pmatrix} q_x \\ q_y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} a_{11} p_x + a_{12} p_y \\ a_{21} p_x + a_{22} p_y \end{pmatrix}$. If $|\mathbf{q}| = |\mathbf{p}|$, then $|\mathbf{q}|^2 = |\mathbf{p}|^2$. Therefore,

$$
\begin{aligned}
q_x^2 + q_y^2 &= \left(a_{11} p_x + a_{12} p_y\right)^2 + \left(a_{21} p_x + a_{22} p_y\right)^2 \\
&= (a_{11}^2 + a_{21}^2) p_x^2 + (a_{12}^2 + a_{22}^2) p_y^2 + 2(a_{11} a_{12} + a_{21} a_{22}) p_x p_y \,. \\
&= p_x^2 + p_y^2
\end{aligned}
$$

The conditions for this to be true are $a_{11}^2 + a_{21}^2 = a_{12}^2 + a_{22}^2 = 1$ and $a_{11} a_{12} + a_{21} a_{22} = 0$, which are exactly the same as the conditions on the elements of $\mathbf{O}$ in part (c).

We have show that, if $\mathbf{q} = \mathbf{A}\mathbf{p}$ and $|\mathbf{q}| = |\mathbf{p}|$ then $\mathbf{A}$ is an orthogonal matrix.

(ii) Since $\mathbf{p}^t = \begin{pmatrix} p_x & p_y \end{pmatrix}$ is an 1×2 matrix and $\mathbf{A}^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$ is an 2×2 matrix, only the matrix product in (2) is well-defined and indeed

$\begin{pmatrix} q_x & q_y \end{pmatrix} = \begin{pmatrix} p_x & p_y \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \Leftrightarrow \mathbf{q}^t = \mathbf{p}^t \mathbf{A}^t$.

(e) $\mathbf{q}_1 \bullet \mathbf{q}_2 = \mathbf{q}_1^t \mathbf{q}_2 = (\mathbf{A}\mathbf{p}_1)^t \mathbf{A}\mathbf{p}_2 = \mathbf{p}_1^t \mathbf{A}^t \mathbf{A}\mathbf{p}_2 = \mathbf{p}_1^t \left(\mathbf{A}^t \mathbf{A}\right)\mathbf{p}_2$, using the result of part (d)(ii).

$\mathbf{p}_1 \bullet \mathbf{p}_2 = \mathbf{p}_1^t \mathbf{p}_2 = \mathbf{p}_1^t \mathbf{I} \mathbf{p}_2$.

Therefore, $\mathbf{q}_1 \bullet \mathbf{q}_2 = \mathbf{p}_1 \bullet \mathbf{p}_2$ if and only if $\mathbf{A}^t \mathbf{A} = \mathbf{I}$.

We have shown that if $\mathbf{q}_i = \mathbf{A}\mathbf{p}_i, i = 1,2$ and $\mathbf{q}_1 \bullet \mathbf{q}_2 = \mathbf{p}_1 \bullet \mathbf{p}_2$ then $\mathbf{A}$ is an orthogonal matrix.

(f) Yes. The column vectors are normalised since $\cos^2\theta + \sin^2\theta = 1$ and they are orthogonal since $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \bullet \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$. Also, we can evaluate directly

$$\mathbf{R}_\theta^t \mathbf{R}_\theta = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$
$$= \begin{pmatrix}\cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta\end{pmatrix}$$
$$= \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}.$$

(g) $\mathbf{O}_1 = \begin{pmatrix}3/5 & 4/5 \\ -4/5 & 3/5\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ with $\theta = -53.13^\circ$ which represents an a *clockwise* rotation of 53.13° of the plane about the origin.

$\mathbf{O}_2 = \begin{pmatrix}3/5 & -4/5 \\ -4/5 & -3/5\end{pmatrix} = \begin{pmatrix}3/5 & 4/5 \\ -4/5 & 3/5\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} = \mathbf{O}_1\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$ which represents a reflection in the *x*-axis $(y \to -y)$ followed by a 53.13° clockwise rotation.

(h) $\mathbf{t}_1 = \mathbf{O}_1\mathbf{s} = \begin{pmatrix}3/5 & 4/5 \\ -4/5 & 3/5\end{pmatrix}\begin{pmatrix}-5 \\ 7\end{pmatrix} = \begin{pmatrix}13/5 \\ 41/5\end{pmatrix}$ and $\mathbf{t}_2 = \mathbf{O}_2\mathbf{s} = \begin{pmatrix}3/5 & -4/5 \\ -4/5 & -3/5\end{pmatrix}\begin{pmatrix}-5 \\ 7\end{pmatrix} = \begin{pmatrix}-43/5 \\ -1/5\end{pmatrix}$.

We find $|\mathbf{s}| = \sqrt{(-5)^2 + 7^2} = |\mathbf{t}_1| = \sqrt{(13/5)^2 + (41/5)^2} = |\mathbf{t}_2| = \sqrt{(-43/5)^2 + (-1/5)^2} = \sqrt{74}$, that is, that magnitude of all three vectors is $\sqrt{74}$.

We evaluate the dot-product: $\hat{\mathbf{s}} \bullet \hat{\mathbf{t}}_1 = \dfrac{\mathbf{s} \bullet \mathbf{t}_1}{|\mathbf{s}||\mathbf{t}_1|} = \dfrac{\mathbf{s} \bullet \mathbf{t}_1}{74} = \dfrac{-65/5 + 287/5}{74} = 0.6 = \cos\theta_{\mathbf{s},\mathbf{t}_1}$ which yields an angle between $\mathbf{s}$ and $\mathbf{t}_1$ $\theta_{\mathbf{s},\mathbf{t}_1} = 53.13^\circ$. Similarly, we find that

$\hat{\mathbf{s}} \bullet \hat{\mathbf{t}}_2 = \dfrac{\mathbf{s} \bullet \mathbf{t}_2}{|\mathbf{s}||\mathbf{t}_2|} = \dfrac{\mathbf{s} \bullet \mathbf{t}_2}{74} = \dfrac{215/5 - 7/5}{74} = 0.5622 = \cos\theta_{\mathbf{s},\mathbf{t}_2}$ which corresponds to an angle between $\mathbf{s}$ and $\mathbf{t}_2$ of $\theta_{\mathbf{s},\mathbf{t}_2} = 55.79^\circ$.

Relative to the positive *x*-axis, the vector **s** lies at +125.54° (anti-clockwise), the vector $\mathbf{t}_1$ lies at +72.41° (anti-clockwise), and the vector $\mathbf{t}_2$ lies at +181.33° (anti-clockwise) or -178.67° (clockwise). The latter can be obtained by reflection of **s** in the *x*-axis to produce $\begin{pmatrix}-5 \\ -7\end{pmatrix}$ followed by a 53.13° clockwise rotation. Please draw the position vectors in a diagram yourself and I will then save part of a tree by not having to copy another page ☺!