Classwork 5 – Discover the Orthogonal Matrix

Definition: A unit vector $\hat{\mathbf{a}} \in \mathbb{R}^n, |\hat{\mathbf{a}}| = 1$ is called a *normalised* vector. Two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ that are perpendicular to each other, $\mathbf{a} \bullet \mathbf{b} = 0$, are called *orthogonal*, and two unit vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}} \in \mathbb{R}^n$ that are perpendicular to each other, $\hat{\mathbf{a}} \bullet \hat{\mathbf{b}} = 0$, are called *orthonormal*.

This present classwork, leads you to the definition of an *orthogonal matrix*. Although the questions relate to two-dimensional vectors and 2×2 matrices, the results are valid in $\mathbb{R}^n$.

(a) (i) Show that the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ represented by the matrices $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ are orthonormal if $u_x^2 + u_y^2 = v_x^2 + v_y^2 = 1$ and $u_x v_x + u_y v_y = 0$.

(ii) Show that these conditions can be expressed in the matrix form $\mathbf{u}^t\mathbf{u} = \mathbf{v}^t\mathbf{v} = 1$ and $\mathbf{u}^t\mathbf{v} = \mathbf{v}^t\mathbf{u} = 0$ where $\mathbf{A}^t$ denotes the transpose of the matrix $\mathbf{A}$ (see page 3 on FS7).

(b) Find the unit vector $\hat{\mathbf{a}}$ in the direction of $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, and find two other unit vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2$ that are orthonormal to $\hat{\mathbf{a}}$.

(c) Consider the 2×2 matrix $\mathbf{O}$ made up of the two orthonormal vectors $\mathbf{u}, \mathbf{v}$ from part (a), that is, $\mathbf{O} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$. Show that $\mathbf{O}^t\mathbf{O} = \mathbf{I}$, where **I** is the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A matrix $\mathbf{O}$ satisfying $\mathbf{O}^t\mathbf{O} = \mathbf{I}$ is called an *orthogonal matrix*.

(d) We will now discover some additional properties of orthogonal matrices.

(i) Consider two vectors $\mathbf{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$ with $\mathbf{q} = \mathbf{A}\mathbf{p}$ where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. If the magnitudes of $\mathbf{p}$ and $\mathbf{q}$ are identical, what conditions are imposed on the elements of $\mathbf{A}$?

(ii) Which of the following statements is correct: (1) $\mathbf{q}^t = \mathbf{A}^t\mathbf{p}^t$ or (2) $\mathbf{q}^t = \mathbf{p}^t\mathbf{A}^t$?

(e) Consider two vectors $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^2$ which are transformed by the matrix $\mathbf{A}$ into $\mathbf{q}_1, \mathbf{q}_2$, that is, $\mathbf{q}_1 = \mathbf{A}\mathbf{p}_1$ and $\mathbf{q}_2 = \mathbf{A}\mathbf{p}_2$, respectively. If the transformation $\mathbf{A}$ does not change the scalar product, that is, $\mathbf{p}_1 \bullet \mathbf{p}_2 = \mathbf{q}_1 \bullet \mathbf{q}_2$ or, in matrix form, $\mathbf{p}_1^t\mathbf{p}_2 = \mathbf{q}_1^t\mathbf{q}_2$, show that $\mathbf{A}^t\mathbf{A} = \mathbf{I}$ in other words that $\mathbf{A}$ is an orthogonal matrix.
The situation in part (d) is the special case where $\mathbf{p}_2 = \mathbf{p}_1$ and $\mathbf{q}_2 = \mathbf{q}_1$.

(f) Is the rotation matrix $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ an orthogonal matrix? Qualify your answer.

(g) Let $\hat{\mathbf{a}}$ and each of the vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2$ from part (b) in turn form two orthogonal matrices $\mathbf{O}_1$ and $\mathbf{O}_2$ like $\mathbf{O}$ in part (c). If the transformation represents a rotation, find the angle. If not, try to figure out what the operation does represent.

(h) Transform the vector $\mathbf{s} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ using each orthogonal matrix from part (g) in turn. Check that the new vectors $\mathbf{t}_1 = \mathbf{O}_1\mathbf{s}$ and $\mathbf{t}_2 = \mathbf{O}_2\mathbf{s}$ have the same magnitude as **s**. Find the angle between **s** and each of the vectors $\mathbf{t}_1, \mathbf{t}_2$, and draw all three vectors on a diagram.