Classwork 4 – Systematic Elimination: Answers

1. The matrix of coefficients and associated determinant is

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 4 \\ 3 & 1 & 2 \end{pmatrix}, \quad \det \mathbf{A} = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -1 & 4 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 5 \\ 4 & 0 & 6 \\ 3 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 5 & 5 \\ 4 & 6 \end{vmatrix} = -10.$$

↑ adding row 3 to row 1 & 2 ↑ expanding by column 2

$$\det \mathbf{B}^{(1)} = \begin{vmatrix} 9 & -1 & 3 \\ 10 & -1 & 4 \\ 6 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 15 & 0 & 5 \\ 16 & 0 & 6 \\ 6 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 15 & 5 \\ 16 & 6 \end{vmatrix} = -10, \text{ using same procedure.}$$

$$\det \mathbf{B}^{(3)} = \begin{vmatrix} 2 & -1 & 9 \\ 1 & -1 & 10 \\ 3 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 15 \\ 4 & 0 & 16 \\ 6 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 5 & 15 \\ 4 & 16 \end{vmatrix} = -20, \text{ using same procedure.}$$

$$\det \mathbf{B}^{(2)} = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 10 & 4 \\ 3 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -11 & -5 \\ 1 & 10 & 4 \\ 0 & -24 & -10 \end{vmatrix} = -1 \begin{vmatrix} -11 & -5 \\ -24 & -10 \end{vmatrix} = 10.$$

↑ adding $(-2)\times$ row 2 to row 1, adding $(-3)\times$ row 2 to row 3 ↑ expanding by column 1

Hence, Cramer's Rule yields $x = \frac{-10}{-10} = 1$, $\quad y = \frac{10}{-10} = -1$, $\quad z = \frac{-20}{-10} = 2$.

Substituting into the original equations we find they are satisfied:

$$2\cdot 1 - (-1) + 3\cdot 2 = 9$$
$$1 - (-1) + 4\cdot 2 = 10$$
$$3\cdot 1 + (-1) + 2\cdot 2 = 6.$$

2. Following the steps specified, we find

$$\begin{aligned} x - y + 4z &= 10 \\ 2x - y + 3z &= 9 \\ 3x + y + 2z &= 6 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x - y + 4z &= 10 \\ y - 5z &= -11 \\ 4y - 10z &= -24 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x - y + 4z &= 10 \\ y - 5z &= -11 \\ 10z &= 20 \end{aligned}$$

↑ Steps (b) & (c) ↑ Step (d)

From Eq.(3) we find $z = 2$, which substituted into Eq.(2) yields $y = -1$, and hence from Eq.(1) we find $x = 1$, that is, $(x, y, z) = (1, -1, 2)$.

3. We apply Gauss elimination:

$$\begin{matrix} x+2y+z=7 \\ -2x+3y-z=-5 \\ 3x+12y-6z=9 \end{matrix} \Leftrightarrow \begin{matrix} x+2y+z=7 \\ 7y+z=9 \\ 6y-9z=-12 \end{matrix} \Leftrightarrow \begin{matrix} x+2y+z=7 \\ y-1.5z=-2 \\ 7y+z=9 \end{matrix} \Leftrightarrow \begin{matrix} x+2y+z=7 \\ y-1.5z=-2 \\ 11.5z=23 \end{matrix}$$

↑ Adding $2\times$ Eq.(1) to Eq.(2) & $(-3)\times$ Eq.(1) to Eq.(3).
↑ Multiplying Eq.(3) with 1/6 and exchange Eq.(2) & Eq.(3).
↑ Adding $(-7)\times$ Eq(2) to Eq.(3)

Eq. (3) yields $z=2$. Substituting into Eq.(2) we find $y=1$ and finally from Eq.(1) $x=3$.

Hence, there is a unique solution, namely $(x,y,z)=(3,1,2)$. We easily check that this indeed is a solution by substituting into the original system of equations.

4. We use Gauss elimination. This time, rather than writing the equations with the unknown variables, we apply the elementary operations on the equation on matrix form:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 6 & 1 \\ 1 & 3 & 0 & 5 \\ 3 & 5 & 2 & 4 \end{pmatrix}\mathbf{x}=\begin{pmatrix} 18 \\ -3 \\ 24 \\ 40 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 4 & -5 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & -1 & -5 \end{pmatrix}\mathbf{x}=\begin{pmatrix} 18 \\ -39 \\ 6 \\ -14 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 0 & 4 & -5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & -3 \end{pmatrix}\mathbf{x}=\begin{pmatrix} 6 \\ -39 \\ 6 \\ -8 \end{pmatrix}$$

↑ Adding $(-2)\times$ Eq.(1) to Eq.(2); adding $(-1)\times$ Eq.(1) to Eq.(3) and adding $(-3)\times$ Eq.(1) to Eq.(4).
↑ Adding $(-2)\times$ Eq.(3) to Eq.(1) and Eq.(3) to Eq.(4)

Now adding $2\times$Eq.(4) to Eq.(2) and interchange Eq.(2), Eq.(3) and Eq.(4) to obtain:

$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -11 \end{pmatrix}\mathbf{x}=\begin{pmatrix} 6 \\ 6 \\ -8 \\ -55 \end{pmatrix} \Leftrightarrow$$

$$\begin{aligned} x_1 \quad +3x_3-x_4 &= 6 \\ x_2-x_3+2x_4 &= 6 \\ -2x_3-3x_4 &= -8 \\ -11x_4 &= -55 \end{aligned}$$

from where we find $x_4=5, x_3=-\frac{7}{2}, x_2=-\frac{15}{2}, x_1=\frac{43}{2}$.

5. The third equation is $3\times$ the first and so describes the same plane. Hence, the system has no unique solution. (Indeed, the determinant of the coefficients is zero because the first row is proportional to the third row.) Therefore, Eq.(1) or Eq.(3) is redundant. Let us remove Eq.(3), add $(-2)\times$Eq.(2) to Eq.(1) and exchange the equations to obtain:

$$\begin{aligned} x-y+4z&=10 \\ y-5z&=-11 \end{aligned}$$ and by adding Eq.(2) to Eq.(1) we find $$\begin{aligned} x-z&=-1 \\ y-5z&=-11 \end{aligned}.$$

On vector form, we have $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -11 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \equiv \mathbf{r}_0 + \lambda\mathbf{d}$, where λ is a real number. This is a line passing through $\mathbf{r}_0$ along the direction $\mathbf{d}$. If you prefer, isolate λ to find $\lambda = z = \frac{y+11}{5} = x+1$. Admittedly, you weren't asked to obtain this result, but who can resist. ☺

6. We evaluate the determinant of the matrix of coefficients:

$$\begin{vmatrix} 1 & 3 & -1 \\ 8 & 9 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 12 & 21 & 0 \\ 4 & 7 & 0 \end{vmatrix} = -1\begin{vmatrix} 12 & 21 \\ 4 & 7 \end{vmatrix} = 0.$$

↑ Adding $4\times$ row 1 to row 2
Adding $2\times$ row1 to row 3

↑ Expanding by column 3

Hence, according to Cramer's rule, there is no unique solution. We apply Gauss elimination to reveal whether there are no solution or infinitely many solutions.

$$\begin{aligned} x+3y-z&=6 \\ 8x+9y+4z&=21 \\ 2x+y+2z&=3 \end{aligned} \Leftrightarrow \begin{aligned} x+3y-z&=6 \\ -15y+12z&=-27 \\ -5y+4z&=-9 \end{aligned} \Leftrightarrow \begin{aligned} x+\frac{7}{5}z&=\frac{3}{5} \\ y-\frac{4}{5}z&=\frac{9}{5} \end{aligned}$$

↑ Adding $(-8)\times$ Eq.(1) to Eq.(2)
Adding $(-2)\times$ Eq.(1) to Eq.(3)

↑ Removing Eq.(2). Multiplying new Eq.(2) by $-\frac{1}{5}$
Adding $(-3)\times$ Eq.(2) to Eq.(1)

Hence we find that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/5 \\ 9/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7/5 \\ 4/5 \\ 1 \end{pmatrix} \equiv \mathbf{r}_0 + \lambda\mathbf{d}, \lambda \in \mathbb{R}$. There are infinitely many solutions and they are on a line passing throught $\mathbf{r}_0$ along the direction specified by $\mathbf{d}$.