Classwork 4 – Cramer's Rule & Gauss Elimination

This classwork is about solving a system of linear equations using either Cramer's rule (only applicable for n equations with n unknowns) or Gauss elimination (always applicable). See overleaf for both methods.

First consider the system of linear equations

$$2x - y + 3z = 9 \qquad \textbf{\textit{(1)}}$$
$$x - y + 4z = 10 \qquad \textbf{\textit{(2)}}$$
$$3x + y + 2z = 6. \qquad \textbf{\textit{(3)}}$$

1. Use Cramer's rule to find $x, y, z.$ Apply the properties of determinants given on Fact Sheet 6 to facilitate their evaluation. Substitute the values of $x, y,$ and z back into the original equations to verify that they are indeed satisfied.

2. Now follow these steps to solve the system of linear equations.
   - (a) Exchange Eqs. (1) and (2).
   - (b) Add $(-3)\times$ (the new) Eq.(1) to Eq.(3) to eliminate *x* from this equation.
   - (b) Similarly, add $(-2)\times$Eq.(1) to Eq.(2) to eliminate *x* from this equation too.
   - (d) Add $(-4)\times$ (the modified) Eq.(2) to Eq.(3) to eliminate *y* from the equation.
   - (e) Hence, obtain *z* from the final version of Eq.(3), *y* from the final version of Eq.(2), and *x* from the final version of Eq.(1).

3. Solve this system of linear equations using Gauss elimination:

$$x + 2y + z = 7 \qquad \textbf{\textit{(1)}}$$
$$-2x + 3y - z = -5 \qquad \textbf{\textit{(2)}}$$
$$3x + 12y - 6z = 9. \qquad \textbf{\textit{(3)}}$$

   Once again, apply a procedure to eliminate *x* from Eq.(2) both *x* and *y* from Eq.(3).

4. Solve this 4×4 system using Gauss elimination:

$$x_1 + 2x_2 + x_3 + 3x_4 = 18$$
$$2x_1 + 4x_2 + 6x_3 + x_4 = -3$$
$$x_1 + 3x_2 + 5x_4 = 24$$
$$3x_1 + 5x_2 + 2x_3 + 4x_4 = 40.$$

5. There's an obvious "problem" with this system of linear equations. Can you see what it is?

$$2x - y + 3z = 9$$
$$x - y + 4z = 10$$
$$6x - 3y + 9z = 27.$$

Solve the system of linear equations.

6. Solve the system of linear equations:

$$\begin{aligned} x + 3y - z &= 6 \\ 8x + 9y + 4z &= 21 \\ 2x + y + 2z &= 3. \end{aligned}$$

Cramer's Rule

Consider a system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$ written on the matrix form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the $n \times n$ matrix of coefficients of the system. If the determinant of $\mathbf{A}$ is different from zero, $\det \mathbf{A} \neq 0$, then there is a unique solution and it is given by

$$x_j = \frac{\det \mathbf{B}^{(j)}}{\det \mathbf{A}} \text{ for } j = 1, 2, \ldots, n,$$

where the matrix $\mathbf{B}^{(j)}$ is the matrix obtained from $\mathbf{A}$ by replacing its j th column with column vector $\mathbf{b}$ making up the right-hand side of the system of equations.

Notice: Cramer's rule is only applicable for a system of n equations with n unknowns. Moreover, Cramer's rule is only applicable when $\det \mathbf{A} \neq 0$, that is, when a unique solution exists. If $\det \mathbf{A} = 0$, no unique solution exists and Cramer's rule does not yield any more information about the system of linear equations.

Gauss elimination

None of the following operations changes the solution of a system of linear equations:

(i) Changing the order of the equations.

(ii) Multiplying all terms in an equation by the same non-zero constant.

(iii) Adding a multiple $r \in \mathbb{R}$ of any equation to any other equation. (The multiple can be negative, so addition includes subtraction.)

The strategy is to leave only x_n in the last equation, only x_{n-1} and x_n in the next last, and so on.