Classwork 3 – Air Traffic Control: Answers

1. Using your right hand, let the index finger point along the positive direction of the x-axis and the middle finger point along the positive direction of the y-axis. In a right-handed coordinate system, the thumb will be pointing along the positive direction of the z-axis. Hence, the z-axis points vertically upwards.

2. (a) For flight PH01, $\mathbf{r}_1 = \mathbf{r}_{01} + \lambda \mathbf{d}_{01}$ is equivalent with the three equations $x_1 = -20 + \lambda$, $y_1 = 20 + 2\lambda$, and $z_1 = 5$; Ignoring the *z*-coordinate and isolating λ we find that $\lambda = x_1 + 20 = \frac{y_1 - 20}{2}$. Similarly, for flight PH02, $\mathbf{r}_2 = \mathbf{r}_{02} + \mu \mathbf{d}_{02}$ is equivalent with the three equations $x_2 = 5 - \mu$, $y_2 = 5 + \mu$, and $z_2 = 7.2 - 0.1\mu$. Ignoring the *z*-coordinate and isolating μ we find that $\mu = 5 - x_2 = y_2 - 5$.

(b) Setting $x_1 = x_2 = x_0$ and $y_1 = y_2 = y_0$, and solving the simultaneous equations $x_0 + 20 = \frac{y_0 - 20}{2}$ and $5 - x_0 = y_0 - 5$ (e.g. by substituting $x_0 = 10 - y_0$ from the second equation into the first to find $30 - y_0 = \frac{y_0 - 20}{2}$ and solving this equation w.r.t. y_0) yields $x_0 = -50/3\,\text{km} \approx -16.7$ km and $y_0 = 80/3\,\text{km} \approx 26.7$ km. See sketch of map on page 2.

(c) Denote the directions of the two flight paths in 2D by $\mathbf{d}_{01}^{2D} = (1,2)$ and $\mathbf{d}_{02}^{2D} = (-1,1)$. Then the angle θ between the two flight paths will be the angle between $\mathbf{d}_{01}^{2D}$ and $\mathbf{d}_{02}^{2D}$. Since the dot-product $\mathbf{d}_{01}^{2D} \bullet \mathbf{d}_{02}^{2D} = 1 \cdot (-1) + 2 \cdot 1 = 1$ and the magnitudes $\left|\mathbf{d}_{01}^{2D}\right| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $\left|\mathbf{d}_{02}^{2D}\right| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ we find $\cos\theta = \frac{\mathbf{d}_{01}^{2D} \bullet \mathbf{d}_{02}^{2D}}{\left|\mathbf{d}_{01}^{2D}\right|\left|\mathbf{d}_{02}^{2D}\right|} = \frac{1}{\sqrt{5}\sqrt{2}} = \frac{1}{\sqrt{10}}$. Solving this equation we find that $\theta = 1.2490\text{ rad} = 71.57°$.

(d) The projection of *any* position vector on a flight path onto a unit normal vector for the flight path yields the distance from the origin (convince yourself that is the case by looking at your plot from 2(b)), that is, $p_i = \mathbf{r}_i \bullet \hat{\mathbf{n}}_i, i = 1,2$. Hence, we have to find a unit normal vector $\hat{\mathbf{n}}_i, i = 1,2$ for each of the two flight paths, that is, unit vectors that are perpendicular to $\mathbf{d}_{01}^{2D}$ and $\mathbf{d}_{02}^{2D}$, respectively. In two dimensions, $(a_x, a_y) \bullet (-a_y, a_x) = 0$ so the two vectors (a_x, a_y) and $(-a_y, a_x)$ are perpendicular. Similarly, (a_x, a_y) and $(a_y, -a_x)$ are perpendicular. Since $\left|\mathbf{d}_{01}^{2D}\right| = \sqrt{5}$ and $\left|\mathbf{d}_{02}^{2D}\right| = \sqrt{2}$, the relevant unit normal vectors for the two flight paths are $\hat{\mathbf{n}}_1 = -\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$ and $\hat{\mathbf{n}}_2 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, respectively. Taking the dot product of each unit normal with <u>*any*</u> position vector on the respective paths, e.g. $\mathbf{r}_{0i}, i = 1,2$ yields $p_1 = \mathbf{r}_{01} \bullet \hat{\mathbf{n}}_1 = -20 \cdot \frac{-2}{\sqrt{5}} + 20 \cdot \frac{1}{\sqrt{5}} = \frac{60}{\sqrt{5}} = 26.8\text{ km}$ and for the flight path of PH02 $p_2 = \mathbf{r}_{02} \bullet \hat{\mathbf{n}}_2 = 5 \cdot \frac{1}{\sqrt{2}} + 5 \cdot \frac{1}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.1\text{ km}$.

3. (a) Since $\mathbf{d}_{02} = -\mathbf{i} + \mathbf{j} - 0.1\mathbf{k}$, when flying a distance of $\sqrt{(-1)^2 + 1^2}$ km = $\sqrt{2}$ km in the x-y plane, the descent is 0.1 km. Hence, the angle of descent δ satisfies $\tan\delta = 0.1/\sqrt{2}$, implying $\delta \approx 0.07\text{rad} \approx 4.04^{\circ}$.

(b) The height of plan PH01 is always $z_1 = 5\text{km}$ since it is flying level. Now we want to determine z_2 at the point where the planes cross, that is, at $(x_0, y_0) = (-50/3, 80/3)$. We find $\mu = (5 + 50/3)\text{km}=65/3\text{km}$ yielding $z_2 = 7.2 - 0.1\mu = 7.2 - 6.5/3 \approx 5.033$ km. The vertical separation $|z_2 - z_1| \approx 0.033\text{km}=33\text{m}$.

(c) To avoid a near miss (or worse), the controller should either instruct PH01 to drop to a lower altitude, or tell PH02 to delay its descent.

(d) The altitude of PH01 is 5 km, so the distance in this case is $\sqrt{p_1^2 + 5^2} = \sqrt{745} = 27.3$ km. For PH02, the direction of the perpendicular is affected, albeit very slightly, by the downward slant of the flight path. However, if one ignores this subtlety, an excellent approximation can be obtained by assuming that the (*x*, *y*) coordinates of the point of closest approach are the same as in 2D. Since the altitude is 7.2 km at this point, one obtains $\sqrt{p_2^2 + (7.2)^2} = \sqrt{101.84} = 10.09$ km for the distance. (An exact calculation shows that this answer is accurate to about 0.1%.)

(e) The angle θ is changed, but only very slightly. Following the procedure applied in solving question 2(c) but now in $\mathbb{R}^3$ we find $\mathbf{d}_{01} \bullet \mathbf{d}_{02} = 1\cdot(-1) + 2\cdot 1 + 0\cdot(-0.1) = 1$, $|\mathbf{d}_{01}| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$, and $|\mathbf{d}_{02}| = \sqrt{(-1)^2 + 1^2 + (-0.1)^2} = \sqrt{2.01}$, leading to $\cos\theta = 1/\sqrt{10.05}$ which yields $\theta = 1.2499\text{rad}=71.61^{\circ}$.