**Mathematical Techniques II: Algebra** (Physics Year 1 Term 1) **CW2a 05/11/2007**

# *Classwork 2 – Roots of a Complex Number: Answers*

1. (i) $w=(x+iy)(x+iy)=x^2+ixy+iyx+i^2y^2=(x^2-y^2)+i2xy$

   (ii) $\mathrm{Re}(w)=x^2-y^2$ (iii) $\mathrm{Im}(z)=2xy$ (not $i2xy$. The imaginary part is real!)

   (iv) $|w|=\sqrt{(x^2-y^2)^2+(2xy)^2}=\sqrt{(x^2+y^2)^2}=x^2+y^2$. (v) $|z|=\sqrt{x^2+y^2}$.

   (vi) $|w|=|z^2|=|zz|=|z||z|=|z|^2$, using the general result that $|z_1z_2|=|z_1||z_2|$.

2. (i) Since $w=2+i2\sqrt{3}$ we find $|w|=\sqrt{2^2+(2\sqrt{3})^2}=\sqrt{4+12}=4$. Hence $|z|=2$.

   (ii) $\mathrm{Re}(w)=x^2-y^2=2$ and $\mathrm{Im}(z)=2xy=2\sqrt{3}$ implying that $xy=\sqrt{3}$.

   (iii) From $xy=\sqrt{3}$ we deduce $y=\sqrt{3}/x$ (since $x\neq 0$). Substituting into the first equation, we find that $x^2-(\sqrt{3}/x)^2=2$ so by multiplying this equation with $x^2$ and rearranging, we find $x^4-2x^2-3=0$.

   (iv) $x^4-2x^2-3$ is a polynomial of order 4, so according to the fundamental theorem of algebra, the equation $x^4-2x^2-3=0$ has exactly 4 roots. Let $u=x^2$ and the equation reads $u^2-2u-3=0$ with solutions $u=\dfrac{2\pm\sqrt{4-4\cdot 1\cdot(-3)}}{2}=\dfrac{2\pm\sqrt{16}}{2}$,

   that is, $u=x^2=3$ or $u=x^2=-1$ yielding $x=\pm\sqrt{3}$ or $\pm i$.

   (v) We have that *x* is real by definition so the meaningful roots in this situation are $x=\pm\sqrt{3}$ associated with $y=\pm 1$. Hence $z=\sqrt{3}+i$ or $z=-\sqrt{3}-i$.

   (vi) We find that $|z|=\sqrt{(\pm\sqrt{3})^2+(\pm 1)^2}=\sqrt{3+1}=2$ and

   $z^2=(\sqrt{3}+i)(\sqrt{3}+i)=3+i\sqrt{3}+i\sqrt{3}-1=2(1+i\sqrt{3})$ as they should be.

   (vii) See overleaf.

   (viii) In general $z=|z|e^{i\theta}$ where $\theta=\arg(z)$ is the argument of *z*. Solving $\arg(z)=\arctan(y/x)$ in conjunction with the Argand diagram, we find that $\arg(w)=\arctan(y/x)=\pi/3$ such that $w=|w|e^{i\arg(w)}=4e^{i\pi/3}$ and similarly $\arg(z)=\arctan(1/\sqrt{3})=\pi/6,-5\pi/6$ such that $z=2e^{i\pi/6}$ or $z=2e^{-i5\pi/6}$

   (ix) We have $w=4e^{i(\pi/3+2\pi k)}$ where $k\in\mathbb{Z}$ in an integer. Hence $w^{1/2}=z=(4e^{i(\pi/3+2\pi k)})^{1/2}=2e^{i(\pi/6+\pi k)}$. Hence, the two different solutions are $z=2e^{i\pi/6}$ or $z=2e^{-i5\pi/6}$ or on rectangular form

$z = 2(\cos\pi/6 + i\sin\pi/6) = \sqrt{3} + i$ and

$z = 2(\cos(-5\pi/6) + i\sin(-5\pi/6)) = -\sqrt{3} - i$ .

(x) We express $w$ on complex exponential form.

$|w| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$ and the principal value of the argument of $w$ is determined by

$\arg(w) = \arctan(2\sqrt{3}/(-2)) = \arctan(-\sqrt{3}) = 2\pi/3 + 2\pi k, k \in \mathbb{Z}$ so

$w = 4e^{i(2\pi/3+2\pi k)}$. Raising $w$ to the power of ½ we find $w^{1/2} = z = 2e^{i(\pi/3+\pi k)}$. The two different solutions with the principal argument of $z$ are $z = 2e^{i\pi/3}$ and $z = 2e^{-i2\pi/3}$.

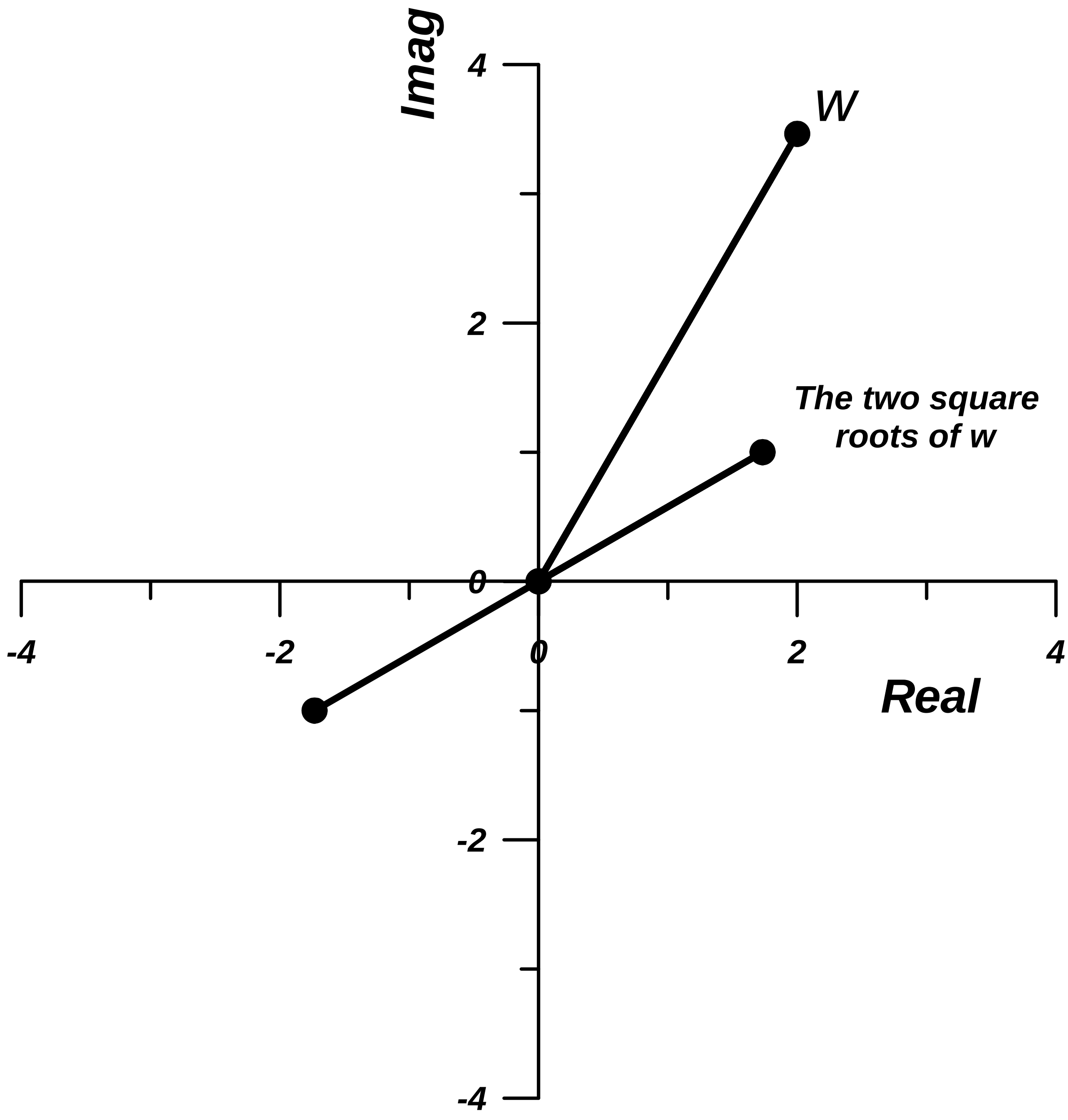
3. (i) We write $8i = 8e^{i\pi/2}$ so raising to the power of 1/3, we find $2e^{i\pi/6}$

(ii) $(2e^{i5\pi/6})^3 = 8e^{i5\pi/2} = 8e^{i\pi/2}e^{i2\pi} = 8i$ so $2e^{i5\pi/6}$ is another cube root of $8i$.

(iii) $(2e^{i9\pi/6})^3 = 8e^{i9\pi/2} = 8e^{i\pi/2}e^{i4\pi} = 8i$ so $2e^{i9\pi/6}$ is yet another cube root of $8i$.

(iv) $2e^{i13\pi/6} = 2e^{i\pi/6}e^{i2\pi} = 2e^{i\pi/6}$ which is the same as the first root found. If the process is repeated further, the results cycle for ever through the three roots found in 3(i)-(iii).

(v) We write $8i = 8e^{i(\pi/2+2\pi k)}, k \in \mathbb{Z}$. Raising to the power of 1/3, we find $(8i)^{1/3} = (8e^{i(\pi/2+2\pi k)})^{1/3} = 2e^{i(\pi/6+2\pi k/3)} = 2e^{i(\pi/6+4\pi k/6)} = 2e^{i(1+4k)\pi/6}$. With $k = 0,1,2$ we re-derive the three different cube roots of $8i$ found in 3(i)-(iii)


Imag
4
2
0
-2
-4
W
The two square
roots of w
-4
-2
0
2
4
Real