Mathematical Techniques II: Algebra (Physics Year 1 Term 1) **CW2 02/11/2007**

Classwork 2 – Roots of a Complex Number

If a complex number is raised to a power (even a complex power), it produces another complex number. In this classwork, we begin by exploring the square roots of complex numbers before moving on to the cube roots. These topics will be covered systematically towards the end of the course (lecture 16).

1. Consider $w = z^2$ where $z = x + iy$ is a general complex number with *x* and *y* real, that is, $x, y \in \mathbb{R}$. Write down, in terms of *x* and *y*:

   (i) w (ii) $\mathrm{Re}(w)$ (iii) $\mathrm{Im}(w)$ (iv) $|w|$ (v) $|z|$

   (vi) What is the relationship between $|w|$ and $|z|$?

2. Consider the specific example where $w = z^2 = 2(1 + i\sqrt{3})$. Our aim is to determine z (i.e., determine x and y), the square root of w.

   (i) Find $|w|$ and $|z|$.

   (ii) Obtain two equations involving *x* and *y*, the real and imaginary parts of *z*.

   (iii) By eliminating *y*, show that *x* satisfies the equation $x^4 - 2x^2 - 3 = 0$.

   (iv) How many roots does this equation have, that is, how many values of *x* satisfy it?

   (v) Are all the roots meaningful in this case? For those that <u>*are*</u> meaningful, find the corresponding values of *y*, and hence specify z.

   (vi) Each pair (*x*,y) defines a complex number z, the square root of w. Check that the modulus of each *z* has the value predicted in 2(i), and that the correct value of *w* is recovered if the square is taken.

   (vii) Plot *w* and its roots in an Argand diagram (the Complex plane).

   (viii) Express w and z on complex exponential form, and find their associated principal arguments in radians.

   (ix) Starting from w on complex exponential form, re-derive the two solutions for z, the square root of w by raising w to the power of 1/2. [Hint: Write the argument for w as $\arg(z) = \theta + 2\pi k$ where $k \in \mathbb{Z}$ is an integer and θ the principal value of the argument of z.]

   (x) Using the procedure outlined in (ix), find the square root of w in the case where $w = z^2 = 2(-1 + i\sqrt{3})$.

3. (i) Find one of the cube roots of the imaginary number 8*i*.
   [Hint: Write 8*i* on exponential form with $\arg(8i) = \theta$, the principal value of the argument of $8i$.]

   (ii) Multiply the solution to (i) by $e^{2\pi i/3}$ and cube the result. What do you deduce?

   (iii) Multiply the solution to (ii) by $e^{2\pi i/3}$ and cube the result. What do you deduce?

   (iv) Multiply the solution to (iii) by $e^{2\pi i/3}$ and cube the result. What do you deduce?

   (v) Write $8i$ on exponential form with $\arg(8i) = \theta + 2\pi k, k \in \mathbb{Z}$. Find all the different cube roots of $8i$.