Classwork 1 – Decay: Answers

(a) Since $A(t)$ denoted the population of species A at time t and the decay is proportial to the population, we find that in a time internal Δt, $A(t+\Delta t) - A(t) = -\alpha A(t)\Delta t,$ where α is the decay constant. Rearranging and taking the limit of $\Delta t \to 0$ yields $\frac{dA}{dt} = -\alpha A.$ There are various ways of solving this differential equation. Using the procedure of separating variables, we find $\frac{dA}{A} = -\alpha dt$ and integrating yields $\int \frac{1}{A} dA = -\int \alpha dt$, that is, $\log_e A(t) = -\alpha t + c$ where c is a constant of integration. Taking the exponential function on both sides of this equation, we have $A(t) = e^c e^{-\alpha t}$. Applying the initial condition, we can determined the constant of integration: $A(0) = e^c$, so we eventually find $A(t) = A(0)e^{-\alpha t}$.

(b) Since α is a rate, it has SI-unit s^{-1} and hence αt is dimensionless (as it should be since it appears as an argument of the exponential function!).

(c) (i) $A(t) = A(0)e^{-1} \Leftrightarrow t = \alpha^{-1}$.

(ii) $A(t) = \frac{1}{2}A(0) \Leftrightarrow e^{-\alpha t} = \frac{1}{2} \Leftrightarrow -\alpha t = -\log_e 2 \Leftrightarrow t = \frac{\log_e 2}{\alpha}$.

(d) The population of species B at time t is the population at time $t = 0$ plus the population of species A that has decays into B since $t = 0$, that is,
$B(t) = B(0) + (A(0) - A(t)) = B(0) + A(0)(1 - e^{-\alpha t})$.

(e) Using similar arguments as in (a), we find that $B(t+\Delta t) - B(t) = [\alpha A(t) - \beta B(t)]\Delta t$
Rearranging and taking the limit of $\Delta t \to 0$ yields the differential equation $\frac{dB}{dt} = \alpha A - \beta B$.

Substituting the solution for $A(t)$ into this equation, we arrive at $\frac{dB}{dt} + \beta B = \alpha A(0)e^{-\alpha t}$.

Multiply through by the integrating factor $e^{\beta t}$ and rearranging the LHS we have $\frac{d(Be^{\beta t})}{dt} = \alpha A(0)e^{(\beta-\alpha)t}$. Integrating, using definite integrals (see Fact Sheet 1, page 2) yields $\left[Be^{\beta t}\right]_0^t = \int_0^t \alpha A(0)e^{(\beta-\alpha)t}dt$. Assuming $\alpha \neq \beta$, we find that $B(t)e^{\beta t} - B(0) = \alpha A(0)\frac{(e^{(\beta-\alpha)t} - 1)}{\beta - \alpha}$. Hence, we find that the population of species B at time t: $B(t) = B(0)e^{-\beta t} + \alpha A(0)\frac{(e^{-\alpha t} - e^{-\beta t})}{\beta - \alpha}$.

(f) When $B(0)=0$, the solution simplifies to $B(t)=\alpha A(0)\dfrac{(e^{-\alpha t}-e^{-\beta t})}{\beta-\alpha}$.

(i) If $\alpha \gg \beta$ then $e^{-\beta t} \gg e^{-\alpha t}$ and $\alpha-\beta \approx \alpha$ so we find;

$$B(t)=\alpha A(0)\frac{(e^{-\beta t}-e^{-\alpha t})}{\alpha-\beta} \approx A(0)e^{-\beta t} \quad (t \gg \alpha^{-1})$$

The $e^{-\alpha t}$ term falls to zero (on a timescale of $\sim \alpha^{-1}$) as the population of A decays rapidly into B. Subsequently, $B(t) \approx A(0)e^{-\beta t}$ as B decays more gradually into C.

(ii) If $\alpha \ll \beta$ then $e^{-\beta t} \ll e^{-\alpha t}$ and $\alpha-\beta \approx -\beta$ so we find;

$$B(t) \approx \frac{\alpha A(0)e^{-\alpha t}}{\beta} \equiv \frac{\alpha}{\beta}A(t) \quad (t \gg \beta^{-1}).$$

The $e^{-\beta t}$ term falls to zero (on a timescale of $\sim \beta^{-1}$) as the population of B reaches dynamic equilibrium with that of A. Thereafter, both species decays at essentially the same rate, maintaining a fixed ratio $\dfrac{B(t)}{A(t)}=\dfrac{\alpha}{\beta} \ll 1$.

Refer to the graphs on the separate sheet.

(g) It is not obvious how to apply the solution of part (e) in the case where $\alpha=\beta$. Here are two ways to overcome the problem.

(1) Go back to $\left[Be^{\beta t}\right]_0^t = \int_0^t \alpha A(0)e^{(\beta-\alpha)t}dt$ and set $\alpha=\beta$ to obtain

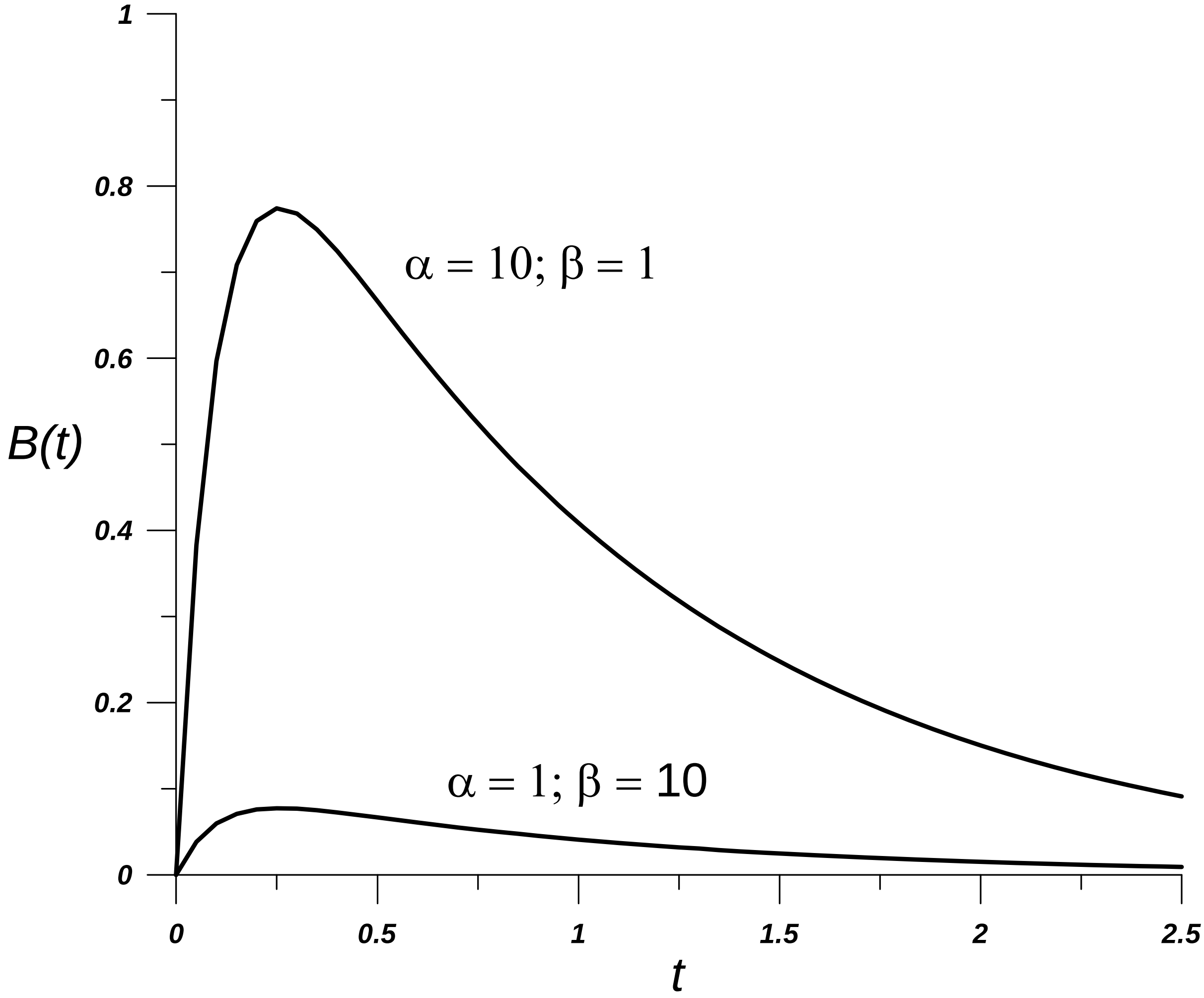
$$B(t)e^{\beta t}-B(0)=\alpha A(0)t \Leftrightarrow B(t)=B(0)e^{-\beta t}+\alpha A(0)te^{-\beta t}.$$

(2) Set $\beta=\alpha+\delta$ and take $\delta \to 0$ at the end of the calculation in (e). From the solution to part (e)

$$B(t)=B(0)e^{-\beta t}+\alpha A(0)\frac{(e^{-\alpha t}-e^{-(\alpha+\delta)t})}{\beta-\alpha}=B(0)e^{-\beta t}+\alpha A(0)e^{-\alpha t}\frac{(1-e^{-\delta t})}{\delta}$$

But $e^{-t\delta}=1-t\delta+\ldots$ for small δ, so

$$B(t)=B(0)e^{-\beta t}+\alpha A(0)e^{-\alpha t}\frac{t\delta}{\delta}=B(0)e^{-\beta t}+\alpha A(0)te^{-\alpha t}$$ as in (1).



B(t)
1
0.8
0.6
0.4
0.2
0
α = 10; β = 1
α = 1; β = 10
0
0.5
1
1.5
2
2.5
t