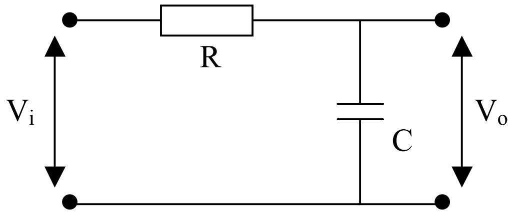
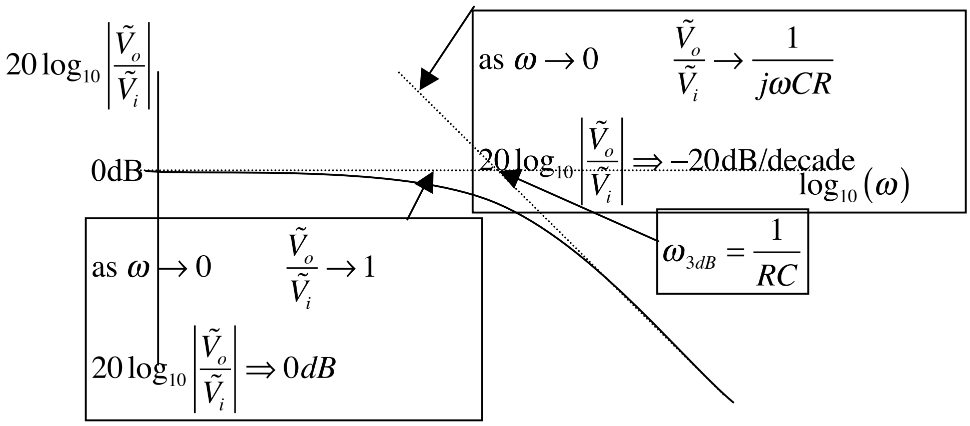
# Electronics problem sheet 4 - solutions, January 2005

**Dr Mark Neil**

1.




$$\frac{\tilde{V}_o}{\tilde{V}_i} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega CR}$$




$$f_{3dB} = \frac{\omega_{3dB}}{2\pi} = \frac{1}{2\pi RC} = 7.95Hz$$

$$\text{At } \omega = 70: \quad \frac{\tilde{V}_o}{\tilde{V}_i} = \frac{1}{1 + j70 \times 10^4 \times 2 \times 10^{-6}} = \frac{1}{1 + j1.4} = \frac{1}{\sqrt{2.96}} e^{-j\tan^{-1}(1.4)}$$

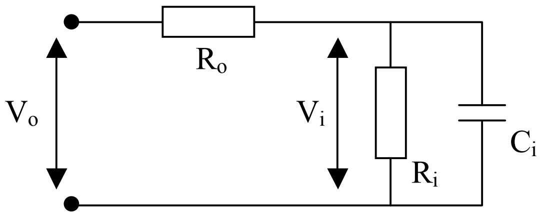
$$\text{At DC}: \quad \frac{\tilde{V}_o}{\tilde{V}_i} = 1$$

$$V_o(t) = \frac{10}{\sqrt{2.96}}\cos\left(70t - \tan^{-1}(1.4)\right) + 15 = 5.81\cos\left(70t - 54.5^\circ\right)$$

2. Removal of 99.9% of 100KHz signal means that 0.1% or -60dB attenuation of signal at 100KHz. As the filter performance will be very close to the -20dB/decade high frequency asymptote then this implies that $f_{3dB}$=100Hz (three decades lower in frequency). A simple RC filter as in question 1 will do and its output impedance (Thevenin or Norton equivalent) is just R in parallel with C. Its maximum value (modulus) is at DC and is therefore simply R so we should choose R=1Ω.

$$f_{3dB} = \frac{\omega_{3dB}}{2\pi} = \frac{1}{2\pi RC} = 100Hz \qquad C = \frac{1}{2\pi R f_{3dB}} = \frac{1}{200\pi} = 1{,}600\mu F$$

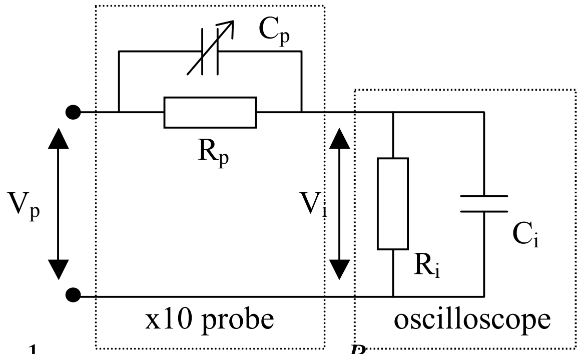
3.




$$\frac{\tilde{V}_i}{\tilde{V}_o} = \frac{R_i // \frac{1}{j\omega C_i}}{R_o + R_i // \frac{1}{j\omega C_i}} = \frac{\frac{R_i}{1 + j\omega C_i R_i}}{R_o + \frac{R_i}{1 + j\omega C_i R_i}} = \frac{R_i}{R_o\left(1 + j\omega C_i R_i\right) + R_i} = \frac{R_i}{R_o + R_i} \frac{1}{1 + j\omega C_i \frac{R_o R_i}{R_o + R_i}}$$

$$\omega_{3dB} = \frac{1}{C_i \frac{R_o R_i}{R_o + R_i}} = \frac{R_o + R_i}{C_i R_o R_i} = 7.8\,Mrads^{-1} \qquad f_{3dB} = \frac{\omega_{3dB}}{2\pi} = 1.24\,MHz$$

4.




$$\frac{\tilde{V}_i}{\tilde{V}_p} = \frac{R_i // \frac{1}{j\omega C_i}}{R_p // \frac{1}{j\omega C_p} + R_i // \frac{1}{j\omega C_i}} = \frac{\frac{R_i}{1 + j\omega C_i R_i}}{\frac{R_p}{1 + j\omega C_p R_p} + \frac{R_i}{1 + j\omega C_i R_i}} = \frac{R_i}{R_p \frac{1 + j\omega C_i R_i}{1 + j\omega C_p R_p} + R_i}$$

if $C_p R_p = C_i R_i$ $\qquad \frac{\tilde{V}_i}{\tilde{V}_p} = \frac{R_i}{R_p + R_i} = \frac{1}{10}$

$R_p = 9R_i \qquad \therefore C_p = \frac{R_i C_i}{R_p} = \frac{R_i C_i}{9R_i} = \frac{C_i}{9} = 1.44\,pF$

at $\omega = 2\pi \times 10^6\, rads^{-1}$

$$\left|Z_{C_p}\right| = \frac{1}{\omega C_p} = 110\,k\Omega \ll 9\,M\Omega \qquad \left|Z_{C_i}\right| = \frac{1}{\omega C_i} = 12\,k\Omega \ll 1\,M\Omega$$

The total input capacitance $C_T$ is approximately equal to $C_p$ in series with $C_i$,

$$C_T = \frac{C_p C_i}{C_p + C_i} = \frac{\frac{C_i}{9} C_i}{\frac{C_i}{9} + C_i} = \frac{C_i}{10} = 1.3\,pF$$

$C_T$ forms a low pass filter in combination with $R_o$=10kΩ from the output of the circuit. Can get 3dB frequency from:

$$f_{3dB} = \frac{\omega_{3dB}}{2\pi} = \frac{1}{2\pi R_o C_T} = 12\,MHz$$

5.

$$Z = Z_R + Z_C + Z_L = R + \frac{1}{j\omega C} + j\omega L = \frac{1 + j\omega CR - \omega^2 LC}{j\omega C}$$

$$|Z| = \frac{\left|1 + j\omega CR - \omega^2 LC\right|}{|j\omega C|} = \frac{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega CR\right)^2}}{\omega C}$$

$$\text{Arg}\left[Z\right] = \text{Arg}\left[1 + j\omega CR - \omega^2 LC\right] - \text{Arg}\left[j\omega C\right] = \tan^{-1}\left(\frac{\omega CR}{1-\omega^2 LC}\right) - \frac{\pi}{2}$$

compare with standard form $\left(1+j\dfrac{\omega}{\omega_o Q} - \dfrac{\omega^2}{\omega_o^2}\right)$

$$\omega_o = \frac{1}{\sqrt{LC}} \qquad f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = \frac{1}{\omega_o RC}$$

$$\tilde{I} = \frac{\tilde{V}_i}{Z} = \frac{j\omega C\tilde{V}_i}{1 + j\omega CR - \omega^2 LC}$$

$$\tilde{V}_R = \tilde{I}R = \frac{j\omega CR}{1 + j\omega CR - \omega^2 LC}\tilde{V}_i$$

6.

$$\frac{\tilde{V}_C}{\tilde{V}_i} = \frac{1}{1 + j\omega RC - \omega^2 LC} = \frac{1}{1 + j\dfrac{\omega}{\omega_o Q} - \dfrac{\omega^2}{\omega_o^2}}$$

$$\left|\frac{\tilde{V}_C}{\tilde{V}_i}\right|^2 = \frac{1}{\left(1 - \dfrac{\omega^2}{\omega_o^2}\right)^2 + \left(\dfrac{\omega}{\omega_o Q}\right)^2}$$

substitute $\quad y = \left|\dfrac{\tilde{V}_C}{\tilde{V}_i}\right|^2, x = \left(\dfrac{\omega}{\omega_o}\right)^2 \quad \Rightarrow \quad y = \dfrac{1}{(1-x)^2 + \dfrac{x}{Q}}$

$$\frac{dy}{dx} = -\frac{-2(1-x) + \dfrac{1}{Q^2}}{\left((1-x)^2 + \dfrac{x}{Q}\right)^2} = -\frac{2x + \left(\dfrac{1}{Q^2} - 2\right)}{\left((1-x)^2 + \dfrac{x}{Q}\right)^2} = 0$$

$$\therefore x = 1 - \frac{1}{2Q^2} \qquad \Rightarrow \qquad \omega = \omega_o\sqrt{1 - \frac{1}{2Q^2}}$$

If $Q^2<0.5$ then the number inside the square root is –ve, the frequency at which the peak occurs would be imaginary and hence there is no peak.

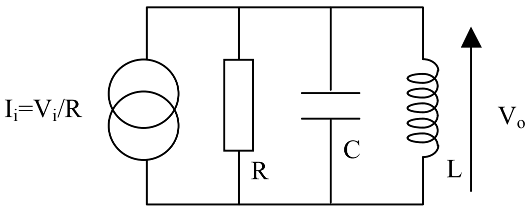
$$Q = \frac{1}{\omega_0 RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$R = \frac{1}{Q}\sqrt{\frac{L}{C}} = \frac{1}{0.5}\sqrt{\frac{4.7\times10^{-3}}{100\times10^{-12}}} = 13.7k\Omega$$

7.

$$Q \approx \frac{\omega_o}{\Delta\omega_{3dB}} = \frac{f_o}{\Delta f_{3dB}} = \frac{10^6}{10^4} = 100$$

Convert suggested arrangement into Norton equivalent




Calculate equivalent impedance $Z_T$ of LCR in parallel

$$Z_T = Z_R//Z_C//Z_L = \frac{1}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}} = \frac{j\omega L}{1 + j\omega\frac{L}{R} - \omega^2 LC}$$

$$\tilde{V}_o = \tilde{I}_i Z_T = \tilde{I}_i \frac{j\omega L}{1 + j\omega\frac{L}{R} - \omega^2 LC} = \tilde{V}_i \frac{j\omega\frac{L}{R}}{1 + j\omega\frac{L}{R} - \omega^2 LC}$$

Transfer function describes a band pass filter (because of ω dependence in numerator) with characteristics obtainable by comparison with the standard form (see above):

$$\omega_o = \frac{1}{\sqrt{LC}} \qquad f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

Choose $C = 100pF$ (for instance)

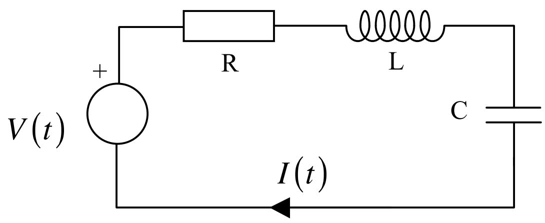
$$L = \frac{1}{4\pi^2 f_o^2 C} = \frac{1}{4\pi^2 100} = 253\mu H$$

$$Q = \frac{R}{\omega_o L} = 100$$

$$R = \omega_o LQ = 2\pi\times10^6 \times 253\times10^{-6}\times100 = 159k\Omega$$

[You would be lucky to achieve a Q as high as 100, in practice the resistances present in the capacitor and inductor would limit the achievable Q and a much higher value of R than that calculated here would be needed]

8.




By applying Kirchhoff's voltage law around the loop we get:

$$V = V_R + V_L + V_C = IR + L\frac{dI}{dt} + \frac{1}{C}\int Idt = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_o\cos(\omega t)$$

For a simple mechanical forced harmonic oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$

| | |
|---|---|
| displacement | $x \Leftrightarrow q$ |
| mass | $m \Leftrightarrow L$ |
| damping constant | $b \Leftrightarrow R$ |
| restoring force constant | $k \Leftrightarrow \frac{1}{C}$ |
| driving force | $F(t) \Leftrightarrow V(t)$ |

9. $V_o$ and $\phi$ essentially depend on the initial conditions, i.e. what is happening in the circuit at time t=0. Two conditions initial conditions are necessary to define $V_o$ and $\phi$ . These might for instance be $V_o$ (0) and $dV_o/dt(0)$ or $V_L(0)$ and $V_R(0)$ or even $V_c(0)$ and I(0).

   For a series resonant circuit

$$\omega_o = \frac{1}{\sqrt{LC}} \qquad Q = \frac{1}{\omega_o RC} = \frac{\omega_o L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

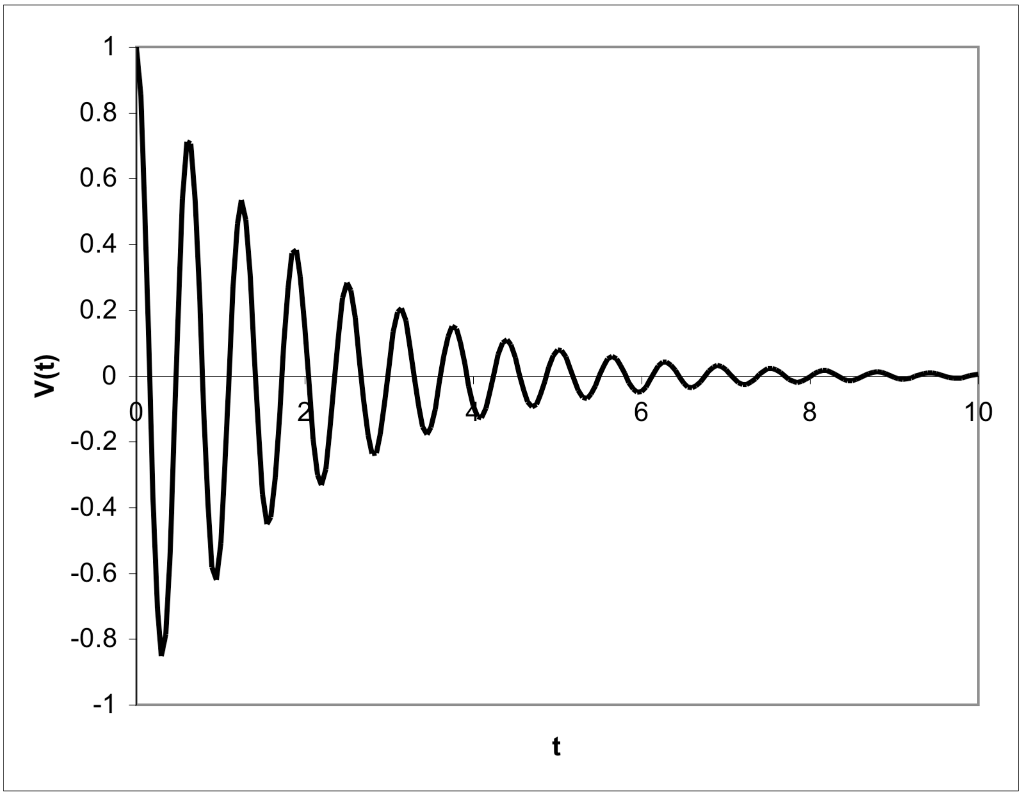
$$\alpha = \frac{\omega_o}{2Q} = \frac{R}{2L} \qquad \omega' = \omega_o\sqrt{1-\frac{1}{4Q^2}} = \sqrt{\omega_o^2 - \frac{\omega_o^2}{4Q^2}} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

   For a parallel resonant circuit

$$\omega_o = \frac{1}{\sqrt{LC}} \qquad Q = \omega_o RC = \frac{R}{\omega_o L} = R\sqrt{\frac{C}{L}}$$

$$\alpha = \frac{\omega_o}{2Q} = \frac{1}{2RC} \qquad \omega' = \omega_o\sqrt{1-\frac{1}{4Q^2}} = \sqrt{\omega_o^2 - \frac{\omega_o^2}{4Q^2}} = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

   Graph below shows of V(t) for the case of $\omega$=10, Q=10, $V_o$=1 and $\phi$=0.


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
V(t)
0
2
4
6
8
10
t