UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2006

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ELECTRICITY & MAGNETISM

For First-Year Physics Students

Monday 5th June 2006: 14.00 to 16.00

Answer ALL parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

SECTION A

1. (i) Electric charges $-q$, q, and $-q$ are placed in a line along the x-axis, at $x = -a$, $x = 0$, and $x = +a$ respectively. Find the electric field at points along the positive x-axis for both $x < a$ and $x > a$.

[6 marks]

(ii) State Gauss's Flux Law, explaining clearly what all the terms and symbols mean. How is Gauss's Flux Law modified in a dielectric material? A spherical capacitor is formed by two concentric conducting shells of negligible thickness and of radii a and $b < a$; the space in between the shells is filled with a dielectric material of relative dielectric constant $\epsilon_r \equiv K = 2.0$ and the outer sphere is grounded. If a charge $+Q$ is placed on the inner sphere, so that $-Q$ appears on the outer sphere, find the electric potential V for $b < r < a$ between the spheres, and hence determine the capacitance C.

[7 marks]

(iii) A long straight conducting cylinder of radius a carries a current I. The current is distributed primarily near the surface of the conductor, so that the current density varies as $j = j_o\ell/\sqrt{a^2 - r^2}$ for $r < a$.

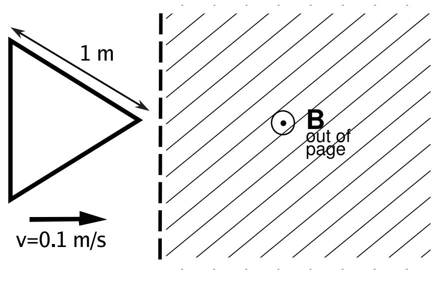
(a) Find the value of j_o.

(b) Use Ampere's Law to find the magnetic field for $r < a$ and $r > a$.

[You may quote $\int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2 - x^2}$]

[8 marks]

(iv) A triangular loop of wire with sides of 1 m is initially outside a region in which there is a uniform magnetic field **B**. One of the edges of the loop is parallel to the boundary of the magnetic field region. The loop has an electrical resistance R. At time $t = 0$ the loop is inserted into the magnetic field at a uniform speed v starting at one vertex (see diagram). The loop is oriented so that all its sides remain perpendicular to **B**.



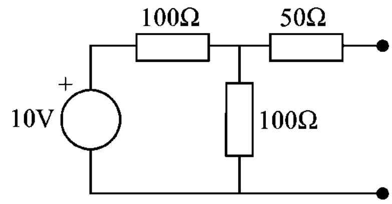
(a) Show that until the loop is totally immersed in the field region the area threaded by **B** increases with time according to $A = (vt)(vt/\sqrt{3})$.

(b) If $|\mathbf{B}| = 1\,\mathrm{T}$, $R = 100\,\Omega$, and $v = 0.1$ m/s, find the current in the loop as a function of time.

[8 marks]

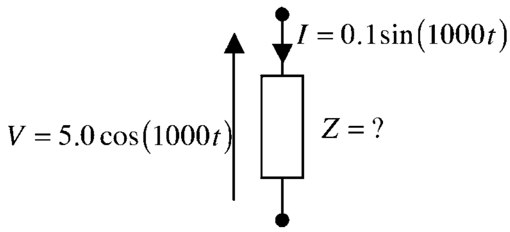
[TOTAL 29 marks]

2. (i) Find the Thévenin and Norton equivalents of the following circuit with respect to the marked nodes.



[7 marks]

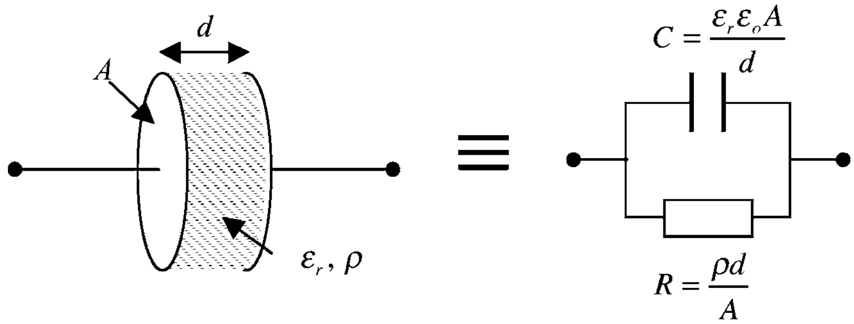
(ii) A voltage $V = 5.0\cos(1000t)$ is applied across a mystery component and a resulting current of $I = 0.1\sin(1000t)$ is measured passing through it.



Write down the voltage and current as complex phasors $\tilde{V}$ and $\tilde{I}$. Hence find the complex impedance Z of the mystery component. Identify the mystery component as a resistor, capacitor or inductor and find its value.

[8 marks]

(iii) An electronic device is constructed with parallel plates of area A and separation d as shown in the diagram. It is filled with a material of relative permittivity ϵ_r and resistivity ρ, such that the capacitance C and resistance R of the device are given by the equations shown, where ϵ_0 is the permittivity of free space.



Show that the time constant, $\tau = RC$, of the resulting parallel resistor capacitor combination is independent of the device dimensions, depending only on the material properties of the filling.

If $\epsilon_r = 3.0$, determine the limits on ρ that ensures $\tau > 1$ sec.

[6 marks]

[TOTAL 21 marks]

SECTION B

3. (see Note at the end of this question concerning notation)

(i) Define electric dipole moment $\mathbf{p}$. An electric dipole is constructed by placing charges $+q$ and $-q$ along the z-axis separated by a distance d, with the charge $+q$ at $+d\hat{\mathbf{z}}$ relative to the charge $-q$. Find the electric dipole moment $\mathbf{p}$ for this configuration.

[4 marks]

(ii) If the dipole given in (i) is located at the origin, use the principle of superposition to calculate the electric potential V at a point with position vector $\mathbf{r}$ and show that, at distances large with respect to d, it reduces to

$$V = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_o r^3}$$

[9 marks]

(iii) This dipole is placed on the z-axis a distance $D \gg d$ above an infinite grounded conductor that occupies the $x - y$ plane.

(a) Use the method of images to find an expression for the resulting electric potential along the z-axis. [You should treat this in the limit of distances large with respect to d, so that the result in (ii) is applicable, after suitable modification to take account of the fact that the dipole is now not at the origin. Note that the orientation of the dipole is unchanged, with the charge $+q$ at $+d\hat{\mathbf{z}}$ relative to the charge $-q$.]

(b) Describe qualitatively, with the aid of a suitable sketch, how the field lines near the conductor differ from those you would find if the conductor were not present.

[7 marks]

(iv) (a) Find the torque on the electric dipole in (iii).

(b) By differentiating the potential V found in (iii), or otherwise, show that the force on a test charge Q located on the z-axis at $z = 2D$ is $14Qqd/27\pi\epsilon_0 D^3$.

[5 marks]

[TOTAL 25 marks]

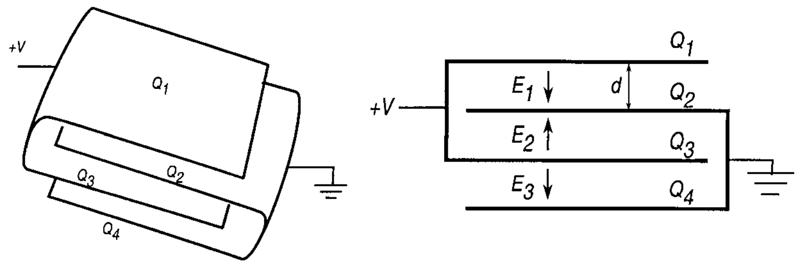
Note: The dipole moment $\mathbf{p}$ *is often denoted* $\mathbf{M_E}$ *and the electric potential* V *is often denoted* ϕ.

4. (i) A parallel plate capacitor is constructed from two square sheets of aluminium foil (a good electrical conductor) separated by a distance $d = 0.15\,\text{mm}$. The area of each foil sheet is $0.5\,\text{m}^2$.

(a) Assuming the space between the sheets is a vacuum and neglecting any edge effects, find the resulting capacitance C.

(b) If the space between the sheets is occupied by a sheet of plastic food wrap with relative dielectric constant $\epsilon_r \equiv K = 4.0$, calculate the capacitance C in this case. [8 marks]

(ii) In an attempt to improve the properties of her capacitors, the owner of Cathy's Capacitor Company experiments with different configurations. In one experiment she constructs a 'sandwich' configuration in which each sheet of conductor, of area A is folded and interleaved with the other, as shown in the following diagram, with a distance d separating each layer.



Treating the space between the sheets as vacuum, use the requirement that the sheets are at the indicated voltages to find the electric field between each pair of sandwich layers (e.g., by application of Gauss's Law or otherwise) and the charges Q_1, Q_2, Q_3, Q_4 on each layer (neglect any edge effects). Hence determine the total capacitance for this system in terms of the area A and separation d. Compare this result with that found in the vacuum case of (i) above to see if Cathy's idea is a good one; i.e., if she can make the same capacitor with less total area of conductor. [8 marks]

(iii) In her final experiment, Cathy makes a cylindrical capacitor of length L with concentric radii b and $a > b$. Show that the capacitance in general is

$$C = \frac{2\pi L \epsilon_o}{\ln(a/b)}.$$

[4 marks]

(iv) Hence show that if the combined area of the two cylinders is A, so that $A = 2\pi L(a+b)$, and there is a fixed vacuum separation distance $d \ll b$ between the concentric cylinders so that $a = b + d$, the resulting capacitance is

$$C \approx \frac{\epsilon_o A}{2d}\left(1 - \frac{d}{2b}\right)$$

[Hint: Expand the logarithmic function and also show that

$$L = \frac{A}{4\pi b}\,\frac{1}{1 + (d/2b)}]$$

[5 marks]

[TOTAL 25 marks]

[You may use $\ln(1 + x) \approx x$ for small x.]

5. The Biot and Savart Law relates a current I along an elemental path $\mathbf{dl}$ to the resulting magnetic field $\mathbf{B}$ at a point with relative position vector $\mathbf{r}$ by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{I\mathbf{dl}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\int \frac{I\mathbf{dl}\times\mathbf{r}}{r^3}$$

(i) An infinite straight wire carrying a current I is located along the z-axis. Use the Law of Biot and Savart to find the magnetic field at the point $(a, 0, 0)$.

[You may quote $\int \left(b^2 + x^2\right)^{-3/2} dx = x/\left[b^2\left(b^2 + x^2\right)^{1/2}\right]$. You may NOT use Ampere's Law to answer this question, although if you wish you could check your answer by applying Ampere's Law.]

[7 marks]

(ii) A particle of electric charge q and mass m is moving under the influence of the Lorentz force $q\mathbf{v}\times\mathbf{B}$ in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

(a) Show that the $\hat{\mathbf{z}}$ component of the particle velocity is a constant. For the rest of this question you should assume that $v_z \equiv 0$.

(b) Write down the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ equations of motion, and hence show that

$$\frac{d}{dt}\left[\frac{1}{2}m(v_x^2 + v_y^2)\right] = mv_x\frac{dv_x}{dt} + mv_y\frac{dv_y}{dt} = 0$$

(c) By differentiating the $\hat{\mathbf{x}}$ equation of motion with respect to time and substituting the $\hat{\mathbf{y}}$ equation, show that the x-motion is harmonic with an angular frequency qB/m. By combining this with the result in (ii)-(b) deduce that the particle's motion in the xy-plane is circular at an angular frequency qB/m.

(d) Find an expression for the radius of that circular motion in terms of the speed v of the particle (recall that $v_z \equiv 0$), and show that for $q > 0$ the particle circulates anti-clockwise around $\mathbf{B}$ when viewed by an observer looking along the magnetic field.

[9 marks]

(iii) Consider now the circulating particle as a current loop.

(a) Find the equivalent current I by calculating the average amount of charge passing a given point on the circular orbit per unit time given that the total charge passes once per orbit.

(b) The magnetic moment, $\boldsymbol{\mu}_m$, of a current loop is defined as the product of the area of the loop and current carried by the loop, with the direction of $\boldsymbol{\mu}_m$ defined at the loop centre by applying the right-hand rule to the current. Calculate the magnetic moment $\boldsymbol{\mu}_m$ for the circulating particle.

(c) Given that the magnetic field at the centre of a circular loop of radius b is $\mathbf{B} = \mu_0\boldsymbol{\mu}_m/\left(2\pi b^3\right)$, calculate the magnetic field at the centre of the 'particle loop' giving both the magnitude and direction.

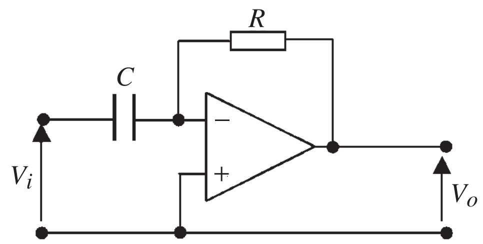
(d) Hence show that the circulating particle is diamagnetic, i.e., it tends to reduce the magnetic field inside its orbit.

[Note: $\boldsymbol{\mu}_m$ is often denoted $\mathbf{M}_B$.]

[9 marks]

[TOTAL 25 marks]

6. An operational amplifier, resistor R and inductor are connected as shown in the diagram with input voltage, V_i, and output voltage, V_0, as marked.



(i) State the following:

(a) The virtual earth approximation as applied to operational amplifier circuits with negative feedback. [4 marks]

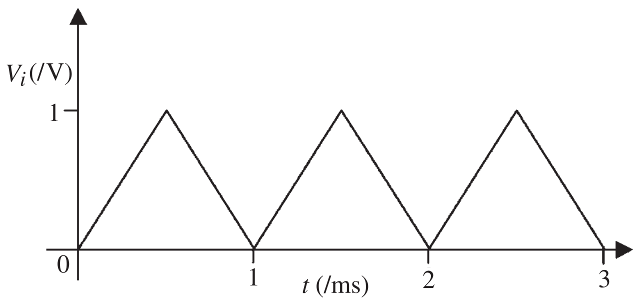
(b) A differential equation describing the relationship between the voltage across a capacitor and the current passing through it. [2 marks]

(ii) By applying Kirchhoff's current law and summing currents at the inverting input to the operational amplifier, or otherwise, show that the output voltage is related to the input voltage by the equation:

$$V_0(t) = -\frac{1}{RC}\frac{dV_i(t)}{dt}$$

[8 marks]

(iii) A 1 V peak-to-peak, 1 kHz, triangular waveform is applied at the input to the circuit as shown in the graph below:



Sketch a graph of the resulting output voltage as a function of time, with appropriate axis scales for the case where $R = 10\,\text{k}\Omega$ and $C = 100\,\text{nF}$. [7 marks]

(iv) How would you rearrange these circuit elements to give an output voltage proportional to the integral of the input voltage? Illustrate your answer with a simple sketch of the resulting circuit. [4 marks]

[TOTAL 25 marks]

End