UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ELECTRICITY & MAGNETISM

For First-Year Physics Students

Wednesday 11th June 2003: 10.00 to 12.00

Answer All parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

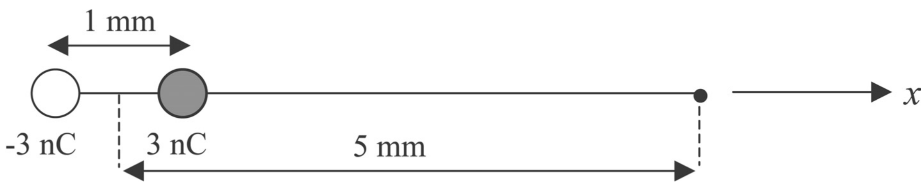
Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) An electric dipole is formed by two charges of $\pm 3\,\text{nC}$ spaced 1 mm apart. Find the magnitude and direction of the electric field on the axis of the dipole at 5 mm from the centre (see diagram).



Find the value of the electric dipole moment p, and use it to compare your answer for the axial field with $E \cong \dfrac{p}{2\pi\varepsilon_0 x^3}$ which is a good approximation at large x. [7 marks]

(ii) State Gauss's Flux Law, explaining what all the terms mean.

Charge is distributed uniformly within a thick spherical shell of inner and outer radii a and b. Use Gauss's Flux Law to show that the field within the shell is

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r^3 - a^3}{b^3 - a^3}\right) \qquad (a < r < b)$$

where Q is the total charge. [7 marks]

(iii) What is the relationship between $\boldsymbol{E}$ (the electric field), $\boldsymbol{D}$ (the electric displacement field) and $\boldsymbol{P}$ (the polarisation field)? What are the units of each?

A uniform electric field of $5 \times 10^3\,\text{V/m}$ exists in a region of empty space. A slab of material of dielectric constant $\varepsilon = 2.5$ is placed so that the electric field is normal to its surface. Find the values of D, P, and E inside the dielectric. [8 marks]

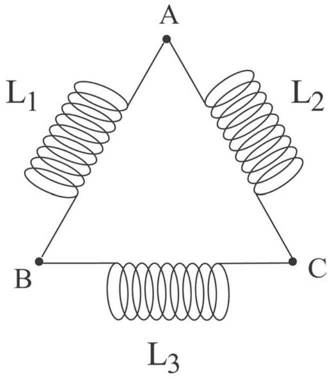
(iv) The Biot-Savart law gives the contribution to the magnetic field at point P from a current element $I\boldsymbol{ds}$ in the form

$$d\boldsymbol{B} = \frac{\mu_0}{4\pi}\,\frac{I\boldsymbol{ds} \times \hat{\mathbf{r}}}{r^2}$$

where r is the distance from the current element to P, and $\hat{\mathbf{r}}$ the unit vector in this direction.

Calculate the magnitude and direction of the field at the centre of a circular loop of wire of radius 0.4 m carrying a current of 5 A. Draw a diagram showing the sense of the field in relation to the direction of the current. [7 marks]

(v) A simple network is constructed from three inductors L_1, L_2 and L_3 as shown below.



(a) How many nodes, branches and loops does the network contain? [2 marks]

(b) Write down a condition on L_1, L_2 and L_3 such that the inductance between any two nodes is independent of the nodes chosen. [2 marks]

(c) If the condition in part (b) holds and $L_1 = 30\,\mu$H, find the inductance that would be measured between the points A and C? [3 marks]

(vi) A simple charge storage device is built from 20 identical $1000\,\mu$F capacitors wired in ***series*** to form a long chain. The chain of capacitors is then charged to 20 kV using an ideal DC power supply connected to the ends of the chain.

(a) What is the total capacitance of the system? [1 mark]

(b) What is the total charge stored in the chain and how much electrical energy does it contain? [2 marks]

(c) At a time $t = 0$ the capacitor chain is abruptly connected across an ideal $100\,\Omega$ resistor. Find the peak current through the resistor and the value of the current flowing after 0.005 seconds. [4 marks]

(vii) A time varying voltage $15\cos(\omega t)$ is applied across a mystery component which has a complex impedance $\mathbf{Z} = 20e^{j\pi/7}\,\Omega$.

(a) Is the mystery component a pure resistor, capacitor or inductor? Give a brief reason for your answer. [1 mark]

(b) Find a simple time domain expression for the current that flows through the mystery component. [3 marks]

(c) How much time averaged power is dissipated in this component? [3 marks]

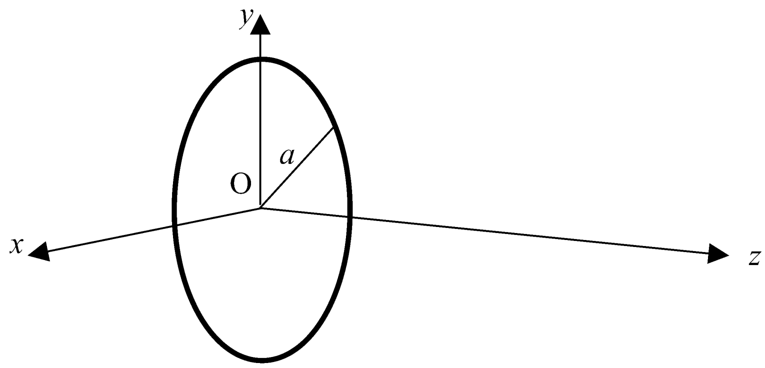
[TOTAL 50 marks]

SECTION B

2. (i) Define electrostatic potential V and electric field $\boldsymbol{E}$. Explain the mathematical relationship between $\boldsymbol{E}$ and V.

[4 marks]

(ii) Consider a thin circular ring of radius a carrying uniform positive charge λ per unit length, and lying in the x-y plane with its centre at the origin O (see diagram).



Show that the electric potential on the z-axis (perpendicular to the plane of the ring) is

$$V(z) = \frac{\lambda}{2\varepsilon_0} \frac{a}{\sqrt{a^2 + z^2}} .$$

Hence find the magnitude and direction of the electric field on the z-axis.

[6 marks]

(iii) Discuss the nature of the equilibrium for a charged particle at O. For what type of charged particle is the equilibrium stable to

(a) small displacements along the z-axis, and

(b) small displacements in the x-y plane?

Can the equilibrium be stable for both kinds of displacement for the same particle?

[4 marks]

(iv) For a particle constrained to move on the z-axis and *stable* to small displacements from O, find an expression for the period of small oscillations about O in terms of the charge and mass of the particle, and the values of λ and a.

[5 marks]

(v) For a particle constrained to move on the z-axis, and *unstable* to small displacements from O, show that the velocity of the particle after release from O is

$$v(z) = \sqrt{u^2 + \frac{\lambda q}{\varepsilon_0 m}\left(1 - \frac{a}{\sqrt{a^2 + z^2}}\right)}$$

where q is the charge, and u is the initial velocity (in the $+z$-direction). What is the terminal velocity?

[6 marks]

[TOTAL 25 marks]

3. (i) Define:

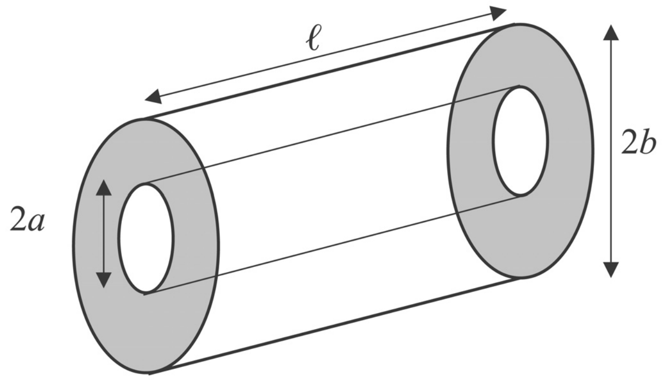
(a) capacitance,

(b) magnetic flux, and

(c) self-inductance.

[5 marks]

(ii) Consider a length ℓ of coaxial cable formed by an inner cylindrical conductor of radius a and a concentric outer conductor of radius b (see diagram). The region between the conductors ($a < r < b$) is filled with an insulating medium of dielectric constant ε. If the outer and inner conductors carry charge densities $+\lambda$ and $-\lambda(\mathrm{C\,m^{-1}})$, find the electric field in ($a < r < b$), and hence the potential difference between them.



[8 marks]

(iii) Show that the capacitance of the length of cable is

$$C = \frac{2\pi\varepsilon_0\varepsilon\ell}{\log_e(b/a)}\,.$$

[4 marks]

(iv) Consider now the situation where axial currents $\pm I$ flow in the conductors. Find the magnitude and direction of the magnetic field B in ($a < r < b$), and hence the inductance of the length of cable.

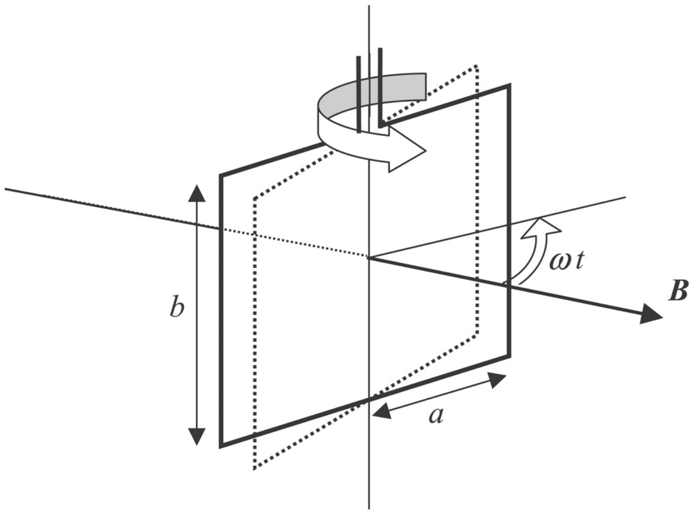
[8 marks]

[TOTAL 25 marks]

4. (i) The force on a charge q moving at velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$ is $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$. Show that the force on an element of wire $d\boldsymbol{s}$ carrying a current I is $d\boldsymbol{F} = I\,d\boldsymbol{s} \times \boldsymbol{B}$ where the direction of $d\boldsymbol{s}$ is in the conventional direction of current flow. [4 marks]

(ii) State Faraday's Law of Electromagnetic Induction. [3 marks]

(iii) In a basic dynamo, a rectangular loop of wire of area $A = 2ab$ is rotated at angular velocity ω about an axis perpendicular to a uniform magnetic field $\boldsymbol{B}$ (see diagram).



If the electrical circuit is completed by a resistor R, show that the magnitude of the current in the loop is

$$I = \frac{BA\omega \cos \omega t}{R}.$$

[6 marks]

(iv) Show that the couple opposing rotation is $\Gamma = IAB \cos \omega t$ and hence verify that the power needed to rotate the loop is the same as the power dissipated in the resistor. [6 marks]

(v) You would like to use the dynamo to supply a typical domestic light bulb (filament resistance $\sim 1\,\mathrm{k}\Omega$) at 50 Hz AC. The maximum magnetic field available is 1 Tesla. On the basis of the formulae above, can the objective be achieved with a realistic loop area? If not, how can you solve the problem? [6 marks]

[The work done by a torque Γ in rotating a body by angle $d\theta$ is $\Gamma d\theta$.]

[TOTAL 25 marks]

5. (i) A simple ***low pass*** filter is constructed from a single resistor R and capacitor C. Make a sketch of the filter circuit and label the input and output terminals. Find an expression for the total impedance of the circuit $\mathbf{Z}_{in}$ between the input and ground and write this in complex exponential form.

[8 marks]

(ii) Write down a simple expression relating the output of the filter $V_{out}(t)$ to the input $V_{in}(t)$ and the voltage drop across the resistor $V_R(t)$.

[1 mark]

(iii) A voltage $V_{in}(t)$ (which might contain both AC and DC components) is fed into the filter. Find a general expression for the voltage drop across the resistor $V_R(t)$. Use this to find an expression for the output of the filter $V_{out}(t)$ that includes both phase and amplitude terms. (Assume that no current is drawn from the filter output).

[7 marks]

(iv) Show that your expression from (iii) gives the correct high and low frequency limits for the magnitude of V_{out}.

[2 marks]

(v) A signal $V_{in} = V_0 + V_1 \cos(\omega t)$ is fed into the filter. If $R = 10\,\mathrm{k}\Omega$ and $C = 1.155\,\mathrm{nF}$, at what frequency in Hz would the peak-to-peak amplitude of the time varying part of V_{out} be half that of V_{in}?

[7 marks]

[TOTAL 25 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

STRUCTURE OF MATTER,VIBRATIONS & WAVES and QUANTUM PHYSICS

For First-Year Physics Students

Monday 9th June 2003: 14.00 to 17.00

Answer ALL questions from Section A, ONE question from Section B, ONE question from Section C and ONE question from Section D.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the SIX answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in SIX answer books even if they have not all been used.

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Turn over for questions

Values of Constants

Acceleration due to gravity	g	$=$	$9.81\,\mathrm{m\,s^{-2}}$
1 Atmosphere		$=$	$1.01 \times 10^5\ \mathrm{N\,m^{-2}}$
Boltzmann's constant	k_B	$=$	$1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Elementary charge	e	$=$	$1.60 \times 10^{-19}\ \mathrm{C}$
Mass of electron	m_e	$=$	$9.11 \times 10^{-31}\ \mathrm{kg}$
Permittivity of vacuum	ε_0	$=$	$8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Planck's constant	h	$=$	$6.63 \times 10^{-34}\ \mathrm{J\,s}$
Planck's constant/2π	$\hbar$	$=$	$1.05 \times 10^{-34}\ \mathrm{J\,s}$
Speed of light	c	$=$	$3.00 \times 10^8\ \mathrm{m\,s^{-1}}$

SECTION A

1. (i) A monatomic ideal gas is allowed to expand adiabatically to twice its original volume. By what factor does the internal energy of the gas change? (For a monatomic ideal gas the ratio of specific heats is $\gamma = 5/3$.)

[3 marks]

(ii) The interaction between a pair of atoms is often approximated by the Lennard-Jones 6-12 potential energy, which has the form:

$$U(r) = \epsilon \left\{ \left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6} \right\}$$

where r is the separation of the atoms (i.e., the distance between their centres). Show that the equilibrium separation is r_0, and sketch a graph of $U(r)$, indicating clearly the position of the equilibrium separation and the value of U at the minimum.

[3 marks]

(iii) A simple model of a gas treats the molecules as rigid spheres of radius a. Show that if only one molecule is moving the average distance it travels between collisions (the mean free path) is:

$$\lambda = \frac{V}{4\pi a^2 N} .$$

[3 marks]

[TOTAL 9 marks]

2. (i) Sketch the variation of the displacement with time of a damped harmonic oscillator in the heavily damped, critically damped and lightly damped regimes. Which regime would be most suitable for (a) a tuning fork, and (b) a shock absorber?

[3 marks]

(ii) The restoring force F of a mechanical oscillator is related to the displacement x and time t by:

$$F = -17x \qquad \text{and} \qquad x = 0.4\cos(8t) ,$$

where all quantities are in SI units. What is the period T? Calculate the maximum velocity v_{max} and maximum potential energy $\mathrm{PE}_{\mathrm{max}}$ of the oscillator. From this calculate the mass m of the oscillator.

[3 marks]

(iii) The wave function of a standing wave on a string is given by:

$$y(x,t) = 0.1\sin(15.71x)\cos(27.2t) ,$$

where all quantities are in SI units. What are the angular frequency, wavevector, amplitude and velocity of the two travelling waves which make up the standing wave? If the string has length 0.8 m, which harmonic is the standing wave?

[3 marks]

[TOTAL 9 marks]

3. (i) Non-relativistic electrons of kinetic energy 100 eV pass through a pair of identical narrow parallel slits separated by 1 μm. Calculate the distance from the central maximum to the nearest zero of the electron interference pattern observed on a screen 1 m from the slits.

[3 marks]

(ii) Write down an expression for the energy levels of a quantum mechanical simple harmonic oscillator in terms of the angular frequency of vibration. When CO molecules make transitions between adjacent vibrational energy levels, photons of wavelength 4.67×10^{-6} m are emitted or absorbed. Find the natural angular frequency of vibration of a CO molecule and evaluate its vibrational zero-point energy in eV.

[3 marks]

(iii) Show that the de Broglie wavelength of a non-relativistic particle of mass m and kinetic energy $3k_BT/2$ is

$$\lambda = \frac{h}{\sqrt{3mk_BT}} \ .$$

The volume per atom in metallic aluminium is $1.66 \times 10^{-29}\ \mathrm{m}^3$ and there are 3 mobile conduction electrons for every atom. By comparing the typical distance between the conduction electrons with their thermal de Broglie wavelength, show that the wave properties of the electrons are important at room temperature.

[3 marks]

[TOTAL 9 marks]

SECTION B

4. (i) Write down the ideal gas equation of state in terms of the pressure, P, the number density, n (i.e., the number of molecules per unit volume) and the temperature, T.

A container of fixed volume V_0 is full of air. There is a hole in the container, and because of this the pressure inside is always equal to the external air pressure. However, the air inside can be kept at a different temperature from the air outside. Assuming that air behaves as an ideal gas, show that if the number density and temperature of the air outside are n_{ex} and T_0 respectively, while the temperature of the air inside is T_1, then the mass of air inside is:

$$m_{in} = \frac{n_{ex} V_0 m T_0}{T_1}$$

where m is the average mass per molecule in air. [4 marks]

(ii) A hot air balloon can be considered as a fixed volume container of the type discussed in part (i). The balloon is required to lift a load of mass m_L (which includes the mass of the material of the balloon itself). Use Archimedes' Principle to show that the balloon will rise if

$$\frac{T_0}{T_1} < 1 - \frac{m_L}{n_{ex} V_0 m} \ .$$

Neglect the volume occupied by the load. [4 marks]

(iii) Assuming that the atmosphere is isothermal, with a temperature T_0, the number density at height h is

$$n(h) = n_0 \mathrm{e}^{-mgh/k_B T_0} \quad \textbf{DO NOT PROVE}$$

where m is again the average mass per molecule in air (assumed to be independent of h). Show that the height reached by a hot air balloon with temperature T_1 is

$$h = \frac{k_B T_0}{mg} \ln \left\{ \frac{n_0 V_0 m}{m_L} \left(1 - \frac{T_0}{T_1} \right) \right\} \ .$$

[6 marks]

(iv) If $T_0 = 290\,\mathrm{K}$ and the pressure at ground level ($h = 0$) is 1 atmosphere, calculate the value of n_0, and, hence, using the expression obtained in part (iii), calculate the height reached by a hot air balloon of volume $V_0 = 10^3\,\mathrm{m}^3$ with $T_1 = 400\,\mathrm{K}$, and a load of mass $m_L = 200\,\mathrm{kg}$. The average mass per molecule in air is $m = 4.82 \times 10^{-26}\,\mathrm{kg}$. [4 marks]

[TOTAL 18 marks]

5. (i) The probability that a molecule in a gas has a speed between v and $v + \mathrm{d}v$ is given by the Maxwell-Boltzmann speed distribution:

$$f(v)\mathrm{d}v = Av^2 \mathrm{e}^{-mv^2/2k_B T}\,\mathrm{d}v$$

where m is the mass of one molecule and A is a constant. Sketch a graph of $f(v)$, indicating v_{mp}, the most probable speed. Show that

$$v_{mp} = \left(\frac{2k_B T}{m}\right)^{1/2} .$$

[5 marks]

(ii) Using the following standard integral (which you may assume without proof)

$$\int_0^{+\infty} x^2 \mathrm{e}^{-\alpha x^2}\,\mathrm{d}x = \frac{1}{4}\left(\frac{\pi}{\alpha^3}\right)^{1/2}$$

show that

$$A = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} .$$

[5 marks]

(iii) Using the following standard integral (which you may assume without proof)

$$\int_0^{+\infty} x^3 \mathrm{e}^{-\alpha x^2}\,\mathrm{d}x = \frac{1}{2\alpha^2}$$

show that the mean speed of the molecules is:

$$\overline{v} = \left(\frac{8k_B T}{\pi m}\right)^{1/2} .$$

[5 marks]

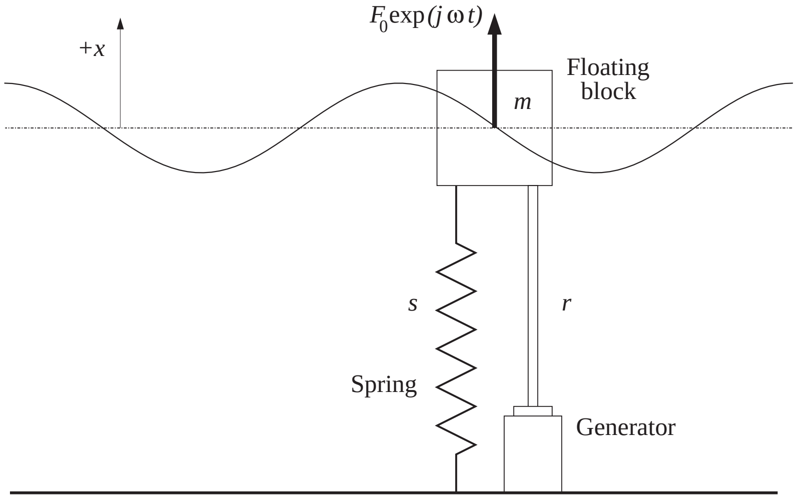
(iv) A thermonuclear fusion reactor contains a fully ionised deuterium plasma at a temperature of 10^8 K. Calculate the mean speed of the ions in the plasma. The mass of a deuterium ion is 3.34×10^{-27} kg.

[3 marks]

[TOTAL 18 marks]

SECTION C

6. A machine is designed to extract energy from ocean waves arriving at the coast. A floating block of total mass m is tethered to the seabed by a spring with spring constant s. A pole extends from the block to the seabed where it drives an electrical generator which damps the block's motion with an effective mechanical resistance r. Surface water waves exert a vertical force on the block of $F_0 \exp(j\omega t)$.



(i) Write down the equation of motion for the vertical displacement $\underline{x}$ of the floating block.
[3 marks]

(ii) Show that the equation of motion has a steady-state solution of the form

$$\underline{x} = \underline{A} \exp(j\omega t) ,$$

where

$$\underline{A} = -\frac{jF_0 \exp(-j\phi)}{\omega Z_m}$$

and

$$Z_m = \sqrt{\left[r^2 + (\omega m - s/\omega)^2\right]} .$$

[5 marks]

(iii) Find the real part of the solution.
[2 marks]

(iv) Using the answer to (iii), show that the resonant frequency ω_r where maximum displacement occurs is given by

$$\omega_r = \sqrt{\omega_0^2 - \frac{r^2}{2m^2}} .$$

What is ω_0?
[2 marks]

(Question 6 continued overleaf)

(v) Sketch the variation of the maximum displacement with ω in the region surrounding ω_r. Show how this varies with r. An engineer wants an approximately constant response to any forces exerted by waves on the machine over a range of frequencies. What advice would you give?

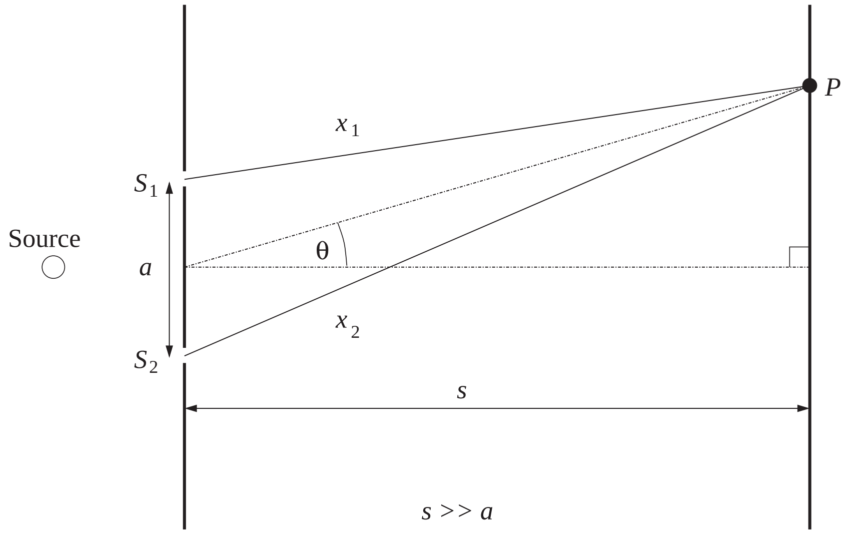
[3 marks]

(vi) The maximum force ever exerted by the waves on the block is 1 kN. The engineer wants the block to operate with a maximum amplitude of 0.5 m at frequencies well below resonance. The engineer also wants the resonant frequency to be equal to 1 Hz and the machine to respond well to all frequencies below 1 Hz. The block weighs 1000 kg. What values of s and r should the engineer use? Calculate the maximum displacement at resonance.

[3 marks]

[TOTAL 18 marks]

7. A coherent source emits electromagnetic waves of a single wavelength λ onto two slits S_1 and S_2 of width d ($\ll \lambda$) separated by a distance a. The slits are positioned a distance s ($\gg a$) from a screen.



(i) At point P, the waves from slits S_1 and S_2 are given by:

$$\psi_1(x_1, t) = A\sin(\omega t - kx_1) ,$$
$$\psi_2(x_2, t) = A\sin(\omega t - kx_2) .$$

Show that the interference maxima occur on the screen at angles

$$\theta_{\text{max}} = \frac{\lambda m}{a} ,$$

where m is an integer. What other phenomenon, apart from interference, would be involved if $d \approx \lambda$.

[5 marks]

(ii) Originally ψ_1 and ψ_2 are in phase. A piece of glass of refractive index n_1 and thickness d is then placed across S_1 so that ψ_1 lags ψ_2 by ϕ. Show that:

$$\phi = \omega d(n_1 - n_2)/c ,$$

where c is the speed of light in vacuum and n_2 is the refractive index of the surrounding medium, which is air.

[4 marks]

(iii) Show that interference maxima now occur at

$$\theta_{\text{max}} = \frac{1}{a}\left[m\lambda + d\left(\frac{n_1}{n_2} - 1\right)\right] .$$

[5 marks]

(iv) Using this experiment, the refractive index n_1 is found to vary with wavelength according to

$$n_1 = C\lambda ,$$

where C is a constant. Calculate the group velocity. What type of dispersion is occurring? Would you expect the intensity of ψ_1 to change relative to ψ_2?

[4 marks]

[TOTAL 18 marks]

SECTION D

8. (i) The Compton scattering formula is

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta) \ ,$$

where h is Planck's constant and c is the speed of light. Identify the physical quantities represented by the symbols λ', λ and m, and sketch a typical Compton scattering event, indicating the scattering angle θ. State the physical assumptions that Compton used in the derivation of this formula.

[6 marks]

(ii) An electron and a positron, both moving at speed $v = \frac{\sqrt{3}}{2}c$ but in opposite directions, collide head on in a particle accelerator, annihilating to produce two photons. Given that electrons and positrons have the same rest mass, show that the two photons have the same energy E, and that $E = 2mc^2$.

[4 marks]

(iii) One of the two photons then strikes the wall of the accelerator, where it is Compton scattered. If the energy after scattering is $E - \Delta E$, where $\Delta E \ll E$, show that

$$\frac{\Delta E}{E} \approx \frac{E}{mc^2}(1 - \cos\theta) \ .$$

How may this equation be simplified when θ is small?

[4 marks]

(iv) If $\theta = 1^o$, calculate both the energy (expressed in terms of mc^2) and the magnitude of the momentum (expressed in terms of mc) transferred to the wall by the scattering event.

[4 marks]

[TOTAL 18 marks]

9. (i) Heisenberg's uncertainty principle states that $\Delta x \Delta p_x \geq \hbar/2$, where Δx and Δp are the root mean square uncertainties in position and momentum, respectively. Assuming that you have been given many identical quantum systems (for example, Hydrogen atoms) in the same quantum state, and that you are able to measure the positions and momenta of the particles involved, describe how you would check this inequality. [4 marks]

(ii) When the uncertainty principle is applied to the electron in a stationary hydrogen atom placed at the origin, $\langle x \rangle$ and $\langle p \rangle$ are zero and hence

$$\Delta x = \sqrt{\langle x^2 \rangle} \quad \text{and} \quad \Delta p_x = \sqrt{\langle p_x^2 \rangle} \, .$$

Given that the root mean square radius $\sqrt{\langle r^2 \rangle}$ of the atom is a, and that the atom is spherical, show that $\Delta x = a/\sqrt{3}$. Assuming minimum uncertainty (i.e., that the equality holds in the uncertainty principle), show that the kinetic energy

$$K = \frac{1}{2} m \langle v_x^2 \rangle + \frac{1}{2} m \langle v_y^2 \rangle + \frac{1}{2} m \langle v_z^2 \rangle$$

of the electron is approximately $9\hbar^2/(8ma^2)$. [5 marks]

(iii) The total energy E of the atom is approximately $K + V$, where $V = -e^2/(4\pi\varepsilon_0 a)$. By minimising E as a function of a, estimate the natural size a_0 of a hydrogen atom. [4 marks]

(iv) An approximate expression for the pressure P required to compress a cube of solid metallic hydrogen by 50% along each axis is

$$P = -\left.\frac{dE}{d\Omega}\right|_{a=a_0/2} ,$$

where $\Omega = 4\pi a^3/3$ is the atomic volume. Evaluate this pressure in atmospheres. [5 marks]

[TOTAL 18 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION April 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

MECHANICS & RELATIVITY

For First-Year Physics Students

Wednesday 30th April 2003: 10.00 to 12.00

Answer ALL parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

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You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

SECTION A (Compulsory)

[Assume $g = 9.8\,\mathrm{m\,s^{-2}}$]

1. (i) State Newton's Third Law. Briefly explain why the example of the forces acting on a book at rest on a table is not a good illustration of the Third Law. [5 marks]

(ii) A cricket ball is hit upwards from the ground at an angle of $\alpha = 53^\circ$ to the horizontal with a speed of $37\,\mathrm{m\,s^{-1}}$. How long does it take to reach its maximum height and how far has it gone in the horizontal direction in that time? Ignore the effects of air resistance. [4 marks]

(iii) A car of mass 1000 kg is travelling at a constant speed of $10\,\mathrm{m\,s^{-1}}$ on a horizontal road when the engine is delivering 5 kW of power. What is the magnitude of the resistive force? When the power is increased to 15 kW the car is able to maintain the same speed up a hill at an angle θ with the horizontal. Assuming that the resistive force is unchanged, calculate the angle θ. [5 marks]

(iv) Show that if no external forces are present, the motion of a two-particle system of masses m_1 and m_2 can be described by an equation of the form:

$$\mathbf{F}_{1on2} = \mu\ddot{\mathbf{r}}$$

where $\mathbf{F}_{1on2}$ is the internal force on particle 2 due to particle 1. Define μ and $\mathbf{r}$ in this case and explain any assumptions made in the derivation. [5 marks]

(v) A rocket travelling at a constant velocity of $2\,\mathrm{km\,s^{-1}}$ in free space has a payload that is 25% of its mass and the remainder is fuel. If the rocket then accelerates by ejecting fuel at a constant speed of $4\,\mathrm{km\,s^{-1}}$ relative to the rocket, what final velocity can it achieve by burning all its fuel? [4 marks]

(vi) Kepler's Second Law says that "The radius vector of a planet sweeps out equal areas in equal times." By considering the area swept out when the radius vector from the Sun to a planet turns through an angle $d\theta$ in time dt, so that the tangential velocity of the planet is given by

$$v_t = r\frac{d\theta}{dt} ,$$

show that Kepler's Second Law is a consequence of the conservation of angular momentum.

[4 marks]

(vii) Consider an inertial frame S' that moves with a velocity u parallel to the x-axis of a second inertial frame S. The coordinate axes of S' and S coincide at the instant $t = t' = 0$. Two events are observed to occur on the x-axis in S. The space and time intervals between the two events are measured in S to be $\triangle x$ and $\triangle t$ respectively. In S' the corresponding space and time intervals are $\triangle x'$ and $\triangle t'$.

(a) The time-dilation formula relates a time interval measured in S to a time interval measured in S' as follows:

$$\triangle t = \gamma \triangle t'$$

where $\gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}}$, $\beta = u/c$ and c is the speed of light. Under what conditions does the time-dilation formula hold? The lifetime of a muon in a frame in which it is at rest is $2.2\,\mu$s. What is the lifetime, observed from earth, of a muon that is approaching the earth with a speed of $0.9\,c$?

[3 marks]

(b) The length contraction formula relates a distance measured in S' to a distance measured in S as follows:

$$\triangle x = \frac{\triangle x'}{\gamma} .$$

Under what conditions does the length-contraction formula hold? A volume of gas is emitted in a jet at a velocity of $0.5\,c$ during the collapse of a massive star. The number of gas molecules per unit volume measured in the frame in which the volume of gas is at rest is N_{gas}. Give an expression for the number of gas molecules per unit volume when measured from the inertial frame in which the star was stationary before it collapsed.

[6 marks]

[TOTAL 36 marks]

SECTION B

2. A particle of mass m moves along the x-axis in a potential of the form

$$U(x) = \frac{a}{x^2} - \frac{b}{x}$$

where a and b are positive constants.

(i) Sketch the potential for $x > 0$ and find the equilibrium position. Is the equilibrium stable?

[8 marks]

(ii) Find the period of small amplitude oscillations about the equilibrium position in terms of a, b and m.

[9 marks]

(iii) What is the velocity v_0 of the particle in the $+x$ direction at the equilibrium position such that the particle just reaches infinity in the subsequent motion? What will happen to the particle if it starts at the equilibrium position with the same speed v_0 but initially moves in the $-x$ direction?

[7 marks]

(iv) The particle is given an energy boost at an infinite distance from the origin and directed back towards the origin with velocity v_b where:

$$v_b = -\frac{2b}{\sqrt{ma}} \quad x = \infty.$$

Find an expression for the distance of closest approach of the particle to the origin in terms of a and b.

[8 marks]

[TOTAL 32 marks]

3. (i) Briefly explain what is meant when a non-relativistic, two-body interaction is described as *elastic*. [2 marks]

(ii) Define an *inelastic* two-body, non-relativistic interaction in terms of the mechanical energy lost (Q). Name TWO types of scattering processes in which Q is positive. Where has the mechanical energy gone in these two cases? Name TWO types of scattering processes in which Q is negative and mechanical energy is released. Where has the mechanical energy come from in these two cases? [7 marks]

(iii) An ice-hockey puck of mass m travelling with speed u in a straight line (assumed to be the x-axis), strikes an equal-mass puck which is at rest on the ice. The first puck is observed to emerge from the collision at speed v_1 and at an angle of 30° with the x-axis and the recoiling puck emerges with speed v_2 at an angle of $-45°$ with the same axis. Sketch the scattering process. Use momentum conservation in the y- and x-directions to find an expression for the ratio v_1/v_2 and a relationship between v_1 and v_2 and u. Hence find expressions for v_1 and v_2 in terms of u. [12 marks]

(iv) If the initial speed $u = 40\,\mathrm{m\,s^{-1}}$ what are v_1 and v_2? [4 marks]

(v) What is the fractional change in the kinetic energy of the system in the scattering? [5 marks]

(vi) What feature of the scattering angles of the two pucks suggests that the collision is inelastic? [2 marks]

[TOTAL 32 marks]

4. (i) A body of mass m at position $\mathbf{r}$ with respect to the origin O of an inertial frame experiences an arbitrary force $\mathbf{F}$ when its instantaneous momentum is $\mathbf{p}$. Define the *torque* $\boldsymbol{\tau}$ of the force about O and the *angular momentum* $\mathbf{L}$ of the particle about O. Show that the rate of change of the angular momentum is equal to the torque. [5 marks]

(ii) Define a *central force*. Show that if the only force acting on a body is a central force then the angular momentum is conserved. [4 marks]

(iii) What is the magnitude of the gravitational force between two bodies of mass m_1 and m_2 a distance $\mathbf{r}$ apart according to Newton's Law of Gravitation? Find an expression relating the acceleration due to gravity g_M on the surface of the Moon (mass M_M radius R_M) compared with the acceleration due to gravity g_E on the surface of the Earth (mass M_E radius R_E). [5 marks]

(iv) A lunar module is in orbit around the Moon. At a certain instant it is a height R_M above the surface of the moon equal to the radius of the Moon (i.e. its distance from the centre of the Moon is $2R_M$). The rocket motor is then fired instantaneously to move the module into a new orbit about the moon that will result in a landing on the lunar surface. The motors are fired so that the magnitude of the module's velocity, v_0, relative to the Moon becomes

$$v_0 = \sqrt{\frac{GM_M}{2R_M}}$$

where G is the gravitational constant. Show that if the mass of the lunar module is m, then the total energy of the lunar module in this orbit is given by

$$E = -\frac{GmM_M}{4R_M}.$$

[4 marks]

(v) The orbit in part (iv) is designed so that the module will be moving tangential to the lunar surface with speed v_f when the module reaches $r = R_M$. Use conservation of energy to find an expression for v_f in terms of G, M_M and R_M. [7 marks]

(vi) When the rocket motor was fired in part (iv) the direction of vector $\boldsymbol{v_0}$ made an angle α with the instantaneous position vector $\boldsymbol{r}$ of the module. Use angular momentum conservation to find the angle α, if the module reaches the Moon's surface with the tangential speed v_f calculated in part (v). [7 marks]

[TOTAL 32 marks]

5. The Lorentz transformation of energy and momentum between two frames S and S', where S' is moving with velocity u in the $+x$ direction relative to S are:

$$E = \gamma(E' + \beta c p'_x), \quad c p_x = \gamma(c p'_x + \beta E'), \quad c p_y = c p'_y, \quad c p_z = c p'_z$$

where $\gamma = \left(1 - \beta^2\right)^{\frac{1}{2}}$, $\beta = u/c$ and c is the speed of light.

(i) A particle, a, of rest mass m_a is moving parallel to the x-axis in S. Given that the relativistic expressions for the energy, E_a, and momentum, p_a, of the particle are $E_a = \gamma_a m_a c^2$ and $p_a = \gamma_a \beta_a m_a c$, show that the rest mass may be written in terms of the energy and momentum as follows:

$$\left(m_a c^2\right)^2 = E_a^2 - (c p_a)^2$$

where $\gamma_a = \left(1 - \beta_a^2\right)^{\frac{1}{2}}$ and $\beta_a = v_a/c$ and v_a is the speed of a measured in S. [8 marks]

(ii) Show that the expression for the rest mass is covariant, i.e. that it takes the same form in all inertial frames. [8 marks]

(iii) An electron and a positron annihilate at rest to produce a pair of photons. What is the energy carried away by each of the photons? (The rest mass of the electron m_e is equal to that of the positron and takes the value $m_e = 0.5\,\mathrm{MeV}/c^2$). [5 marks]

(iv) Consider now the case in which a high-energy electron annihilates with a high-energy positron. A heavy Z particle is created at rest as a result of the collision and no other particles are produced. The rest mass of the Z is $90\,\mathrm{GeV}/c^2$ (1 GeV is equal to 1000 MeV). At what energy must the electron and positron be brought into collision for this process to occur? [4 marks]

(v) Use the above results to find an expression for the energy a positron would require when colliding with a stationary electron for a Z particle to be created. [7 marks]

[TOTAL 32 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

MATHEMATICAL PHYSICS THEORY

For First-Year Physics Students

Friday 13th June 2003: 10.00 to 12.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Provide exact definitions of:

(a) a limit point,
(b) a closed set,
(c) an open set,
(d) the union of sets,
(e) the intersection of sets,
(f) a dense set and
(g) the complement of a set.

[7 marks]

(ii) For the following sets state if they are closed, open or neither, and prove your result.

(a)
$$\mathcal{A} = \bigcup_{n=1}^{\infty} \left] \frac{1}{1+n^2}, \frac{1}{n^2} \right[$$

(b)
$$\mathcal{B} = \left\{ \frac{1}{n} \middle| \; n \in \mathbb{Z}^+ \right\}$$

(c)
$$\mathcal{C}_N = \bigcap_{n=1}^{N} \left] -\frac{1}{\sqrt{n}}, +\frac{1}{\sqrt{n}} \right[\qquad \text{with } N \in \mathbb{Z}^+$$

[9 marks]

(iii) Consider the limit $\lim_{N\to\infty} \mathcal{C}_N = \mathcal{C}_\infty$. Is C_∞ open, closed or neither? Prove your answer.

[2 marks]

(iv) How do the results for C_N and C_∞ illustrate the general open/closed properties of the intersections of finite and infinite numbers of open/closed sets? You may quote any relevant theorems.

[2 marks]

[TOTAL 20 marks]

2. (i) State the ratio test for the convergence of an infinite series. [2 marks]

(ii) State the integral comparison test for the convergence of an infinite series. [2 marks]

(iii) Given a monotonically falling sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = 0$, prove that

(a) The sequence defined by $r_n = \sum_{k=1}^{2n} (-1)^k a_k$ is monotonically falling and that the sequence $s_n = \sum_{k=1}^{2n+1} (-1)^k a_k$ is monotonically growing.

(b) Show that $-a_1 + a_2 \geq r_n \geq s_n \geq -a_1$.

(c) Now show that $\lim_{n\to\infty} |r_n - s_n| = 0$.

(d) Use this to prove that the sequence defined by $\sum_{k=1}^{n} (-1)^k a_k$ converges.

[6 marks]

(iv) Test the following series for convergence

(a) $$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

(b) $$\sum_{n=0}^{\infty} \frac{n!}{n^n}$$

(c) $$\sum_{n=0}^{\infty} \frac{1}{n \ln^2 n}$$

(d) $$\sum_{n=1}^{\infty} \frac{(n+1)^{n-1}}{(-n)^n}$$

(e) $$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n^2+1}}{n \ln n}$$

[10 marks]

[TOTAL 20 marks]

3. (i) Define the concepts of convexity and concavity for functions. Make a drawing that clarifies these notions.

[4 marks]

(ii) Find the intervals in which the polynomial $p(x) = x^4 + 6x^3 + 12x^2 + 12x + 48$ is convex and those in which it is concave.

[6 marks]

(iii) Prove that the composition of two functions g and f given by $h = g \circ f$ is concave if g is concave and monotonously growing and f is concave.

[5 marks]

(iv) Prove that for any natural number n the function $f_n(x) = x^{(1/2)^n}$ is concave on its domain of definition.
[Hint: First prove that $x^{1/2}$ is concave and then use part (iii)]

[5 marks]

[TOTAL 20 marks]

4. (i) Define domain and image of a function. Define furthermore, what is an injective function, a surjective function and bijective function.

[4 marks]

(ii) Give the precise definition in terms of maps of what is meant when we say that two sets have the same size.

[2 marks]

(iii) Define the terms countably and uncountably infinite.

[2 marks]

(iv) Prove that the set of positive rational numbers is countably infinite.

[4 marks]

(v) Consider the set $\mathcal{P}_1$ of all polynomials of degree 1 with coefficients from the natural numbers, i.e. polynomials of the type $p_{ab}(x) = a + bx$. Prove that the set is countably infinite.

[4 marks]

(vi) Prove that the union of countably many countable sets is countable.

[4 marks]

[TOTAL 20 marks]

5. (i) The Mean Value Theorem states that if $f : \mathcal{I} \to \mathbb{R}$ where $\mathcal{I}$ is a closed real interval, and both f and its derivative f' are continuous in $\mathcal{I}$, then for any $a, b \in \mathcal{I}$ there exists some $c \in \mathcal{I}$ such that $f(b) - f(a) = f'(c)(b - a)$. Demonstrate the validity of this theorem with a sketch.

[3 marks]

(ii) Give a precise definition for

(a) a fixed point,

(b) attractive fixed point and

(c) a repulsive fixed point?

[3 marks]

(iii) Consider $f : \mathcal{I} \to \mathcal{I}, \mathcal{I} = [a, b] \subset \mathbb{R}$, f and f' are continuous in $\mathcal{I}$, and $|f'(x)| < 1 \ \forall x \ \in \mathcal{I}$. This guarantees that f has at least one fixed point in $\mathcal{I}$. Use this and the Mean Value Theorem to show:-

(a) f has exactly *one* fixed point in $\mathcal{I}$.

(b) $|f(x) - f(y)| < |x - y| \ \forall \ x, y \in \mathcal{I}$.

(c) The unique fixed point of f in $\mathcal{I}$ is attractive.

[11 marks]

(iv) Let $f : x \mapsto x^2$. What are the fixed points for $x \in \mathbb{R}$? Find the largest interval, $\mathcal{I}$, to which the result of the previous part can be applied and hence find an attractive fixed point. Sketch this function and its fixed points.

[3 marks]

[TOTAL 20 marks]

6. (i) Define precisely the terms:

(a) eigenvalues,

(b) eigenvectors and

(c) singular values.

[3 marks]

(ii) For a general 2×2 matrix $\mathbf{A}$ show that the two eigenvalues λ_1, λ_2 are given by

$$\lambda_{1/2} = \frac{tr\mathbf{A}}{2} \pm \sqrt{(\frac{tr\mathbf{A}}{2})^2 - det\mathbf{A}}$$

[3 marks]

(iii) Compute eigenvalues, eigenvectors and singular values for the matrices

$$M_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

What do you observe for the relation between eigenvalues and singular values of a matrix.

[8 marks]

(iv) Prove that for a real symmetric 2×2 matrix with distinct eigenvalues, the corresponding eigenvectors are orthogonal.

[6 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ANCILLARY PHYSICS

For First-Year Chemistry Students

Friday 20th June 2003: 9.30 to 12.30

Answer SIX questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the SIX answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in SIX answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Explain what is meant by the term *undamped simple harmonic motion*? [3 marks]

(ii) A mass m suspended from a vertical spring of spring constant k, is at rest at a position x_r. By considering the forces acting on the mass, write down an expression relating the equilibrium extension x_r to the spring constant and the mass. [3 marks]

(iii) The mass is extended by a distance x_1 so that it is at $x = x_r + x_1$. By considering the net force acting on the mass in this new position, use Newton's second law of motion to write down the equation of motion of the mass in terms of x. Making use of the expression found in part (ii), show that the equation of motion may be written in terms of x_1 in the form:

$$\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 = 0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

[6 marks]

(iv) Show that the general solution to the above equation of motion is given by

$$x_1 = A\cos(\omega_0 t + \phi)$$

where the constants A and ϕ are the amplitude and initial phase, respectively. [4 marks]

(v) Determine the constants A and ϕ for a 3 kg mass given that $x_r = 0.2\,\text{m}$ and the mass is released with zero velocity at an initial value of $x_1 = 0.1\,\text{m}$. What is the period of the oscillation? [Use $g = 9.8\,\text{m}\,\text{s}^{-2}$.] [4 marks]

[TOTAL 20 marks]

2. (i) Show that the plane wave $y(x,t) = A\cos(kx - \omega t)$ is a solution of the wave equation:

$$\frac{\omega^2}{k^2}\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}\,.$$

[5 marks]

(ii) Show that a superposition (or summation) of two plane waves of equal amplitude A, but different angular frequencies ω_1 and ω_2, can be given by:

$$y(x,t) = 2A\cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)\,.$$

[4 marks]

(iii) Explain how the above combination of the two plane waves can give rise to the phenomenon of *beats* when the two angular frequencies are only slightly different. Illustrate your answer with a drawing.

[5 marks]

(iv) Derive an expression for the beat frequency f_{beat} in terms of the frequencies of the two waves.

[4 marks]

(v) If two tuning forks, one tuned to a frequency of 500 Hz and the other tuned to a frequency of 557 Hz, are sounded together, what beat frequency will be heard?

[2 marks]

Hint:

$$\cos A = \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

[TOTAL 20 marks]

3. (i) By considering a crystal as a set of partially reflective parallel planes of separation d, derive the Bragg condition for constructive interference of scattered radiation of wavelength λ incident at an angle θ to the crystal planes.

[6 marks]

(ii) Rearrange the Bragg condition to obtain an expression for $\sin\theta$ and sketch the allowed value of $\sin\theta$ for $0 < \lambda/2d < 1$, for $n = 1, 2, 3$ and 4.

[4 marks]

(iii) A crystal sample is illuminated with X-rays and maxima are observed in the scattered radiation at $20.5°$ and $44.5°$. What maxima will be observed if the wavelength of the X-ray is doubled?

[5 marks]

(iv) Sketch the experimental set-up used to make such measurements and explain why X-rays are used?

[5 marks]

[TOTAL 20 marks]

4. Write short notes on any FOUR of the following topics.

(i) The conservation of energy in the simple harmonic oscillator.

(ii) The Doppler effect for both observer and source and give an example.

(iii) X-ray crystallography and state the Bragg condition.

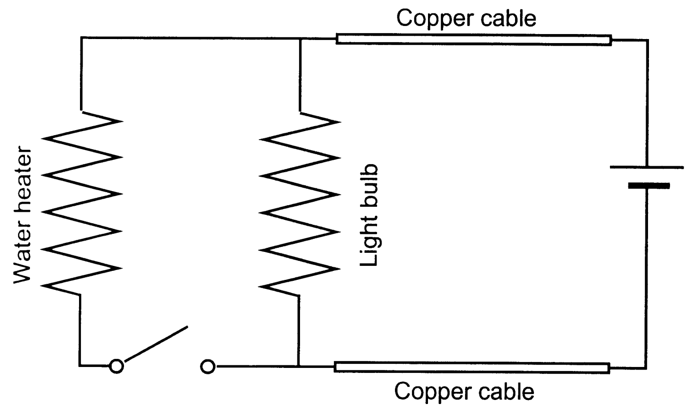
(iv) Resonance effects in forced oscillations.

(v) Standing waves and their relevance to musical instruments.

(vi) Multiple beam interference and its use in an optical spectrometer. Explain the factors influencing resolvance.

[20 marks]

5. In a simple model of the electric connections in a house, a water heater and a light bulb are connected in parallel to the electricity network through two long external copper cables. The electricity network can be regarded as providing an EMF of 240 V.



(i) Write down Kirchoff's rules for electric circuits and relate them to conservation laws.

[4 marks]

(ii) The water heater has an internal resistance giving a 10 kW output at a voltage of 240 V and the light bulb a 40 W output at 240 V. Calculate for both the value of their internal resistance.

[2 marks]

(iii) Each of the connecting copper cables has a radius of 1.5 mm and a length of 100 m. Calculate the resistance of a single cable.

[4 marks]

(iv) Using Kirchoff's rules, or otherwise, calculate the power deposited in the light bulb when the water heater is disconnected. You may assume that the wires inside the house have no resistance. Repeat the calculation when the water heater is connected.

[6 marks]

(v) In the case where the water heater is connected calculate the power deposited per unit length in the connecting copper cable.

[4 marks]

[TOTAL 20 marks]

The resistivity of Copper is $2.65 \times 10^{-8}\,\Omega$ m.

6. (i) State the definition of a dipole moment. Remember to define all symbols used. [2 marks]

(ii) A water molecule can be modelled as an Oxygen atom connected to two Hydrogen atoms with an opening angle of 105° and where the electron clouds of the Hydrogen atoms are pulled slightly toward the Oxygen atom. Make a drawing that clearly indicates how the overall dipole moment of a water molecule is a sum of two parts. [4 marks]

(iii) If the overall dipole moment of a water molecule is 1.85 D, what is the dipole moment of each of the two parts from (ii)? [4 marks]

(iv) Calculate how far a single electron has to move away from the proton in the Hydrogen atom to provide the dipole moment from (iii). What is the force between the electron and the proton at this distance? [6 marks]

(v) Explain why ordinary table salt (NaCl) dissolves easily in water but very poorly in benzene (C_6H_6) which has zero dipole moment. [4 marks]

[TOTAL 20 marks]

$$1\,\mathrm{D} = 3.34 \times 10^{-30}\,\mathrm{C\,m}$$
$$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$$
$$e = 1.60 \times 10^{-19}\,\mathrm{C}$$

7. Write notes, equations and/or make drawings to explain FOUR of the following topics. All symbols used should be defined.

(i) High voltage breakdown.

(ii) Dielectrics.

(iii) Gauss' law.

(iv) Ampère's law.

(v) Magnetic flux.

(vi) The principle of an Ammeter.

[TOTAL 20 marks]

8. (i) Write down Ampère's law and define clearly all the symbols used. [6 marks]

(ii) A very long conductor has a rectangular cross section of width w and height h. It is assumed that the height is much smaller than the width. If a current I flows through the conductor, what is the magnetic field just above the conductor? Make a drawing that shows the field lines and the current I. [6 marks]

(iii) Now a second conductor of the same shape is placed very close to the top of the first one and with a current of the same magnitude I running in the opposite direction. What is the magnetic field in the space between the two plates? Make a drawing that illustrates the field lines between and above/below the conductors. [6 marks]

(iv) If $I = 2.0\,\mathrm{A}$, $w = 5.0\,\mathrm{cm}$ and $h = 1.0 \times 10^{-4}\,\mathrm{m}$, what is the value of the magnetic field between the two plates? [2 marks]

[TOTAL 20 marks]

$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Wb\,A^{-1}\,m^{-1}}$

9. Consider a nucleus of mass number A, atomic number Z and radius R.

(i) The semi-empirical mass formula is based on the liquid drop model. Give two pieces of evidence supporting this. What do they imply for the range and sign of the nuclear force? Give the relationship between R and A.
[5 marks]

(ii) The semi-empirical mass formula allows us to define the total binding energy of the nucleus as:

$$\text{Binding energy} = a_1 A - a_2 A^{2/3} - a_3 \frac{Z^2}{A^{1/3}} - a_4 \frac{(N-Z)^2}{A} \pm \delta.$$

For each term explain its physical origin, the powers of A and Z used, and its sign.
(You may assume a_1, a_2, a_3, a_4 and δ are positive)
[15 marks]

[TOTAL 20 marks]

10. (i) What are alpha, beta and gamma particles?
[2 marks]

(ii) For alpha, beta and gamma decay give:

(a) The effect on a nucleus of ${}^{A}_{Z}\mathrm{X}_{\mathrm{N}}$.
[4 marks]

(b) The energy spectrum of the emitted particle(s).
[3 marks]

(iii) Explain the term half-life, relating it to the decay constant, λ, of a radioactive substance.
[3 marks]

(iv) The ^{238}U decay chain terminates in the stable isotope ^{206}Pb. A sample of rock containing ^{238}U can be dated by assuming that it contained no ^{206}Pb at its formation, and that the half-life of ^{238}U (4.5×10^9 y) is much longer than any of the other half-lives in the decay chain. If the present ratio of ^{238}U to ^{206}Pb is 2, what is the age of the rock?
[8 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ELECTROMAGNETISM & OPTICS

For Second-Year Physics Students

Wednesday 4th June 2003: 10.00 to 12.00

Answer All parts of Section A, TWO questions from Section B and ONE question from Section C.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A (Compulsory)

1. (i) Poynting's theorem is

$$\frac{d}{dt}\int_V \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right) dV + \oint_S \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\cdot d\mathbf{S} = -\int_V \mathbf{J}\cdot\mathbf{E}\, dV$$

where the surface S encloses the volume V and **J** is the total current. Give the physical interpretation of each of the terms in this equation.

[3 marks]

(ii) Derive an expression for the time averaged energy flux from a monochromatic (i.e. single frequency) plane wave.

[3 marks]

(iii) The sun emits about 3.8×10^{26} watts of Electromagnetic radiation and the radius of the sun is 7.0×10^{8} m. Estimate the average electric field of the EM radiation at the surface of the sun.

[2 marks]

(iv) What is the solar radiant energy flux at earth – 1.5×10^{11} metres from the sun. [2 marks]

You will need: $\epsilon_0 = 8.9 \times 10^{-12}\,\mathrm{F\,m^{-1}}$, $\quad c = 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$.

[TOTAL 10 marks]

2. (i) Derive the equation for a thin lens relating the object and image distances to the focal length of the lens.

[3 marks]

(ii) Describe the conditions under which we have Fraunhofer diffraction. [2 marks]

[TOTAL 5 marks]

SECTION B

3. Consider plane electromagnetic waves propagating in the z direction and polarized in the x direction radiating from an oscillating source current $J_x(z,t)$. Maxwell's equations reduce to:

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z}$$

and

$$\frac{1}{c^2}\frac{\partial E_x}{\partial t} = -\frac{\partial B_y}{\partial z} - \mu_0 J_x(z,t).$$

(i) Show that E_x and B_y obey the wave equation in regions where $J_x = 0$. [3 marks]

(ii) Write down travelling wave solutions with frequency $\omega/(2\pi)$. [2 marks]

Let us take the current to be a sheet current oscillating at frequency $\omega/(2\pi)$. Specifically:

$$J_x(z,t) = J_0 \sin(\omega t)\delta(z)$$

where $\delta(z)$ is the Dirac delta function.

(iii) Write down the *form* of the waves in the regions $z > 0$ and $z < 0$ that radiate *from* the current at $z = 0$. You are not expected to have found the amplitude of these waves – yet. [2 marks]

(iv) Show that E_x is continuous across the current sheet and that B_y jumps so that:

$$B_y(\epsilon,t) - B_y(-\epsilon,t) = -\mu_0 J_0 \sin(\omega t)$$

where ϵ is an infinitesimal positive number. [3 marks]

(v) Find the radiated waves [4 marks]

(vi) Suppose J_x has a width Δz (as opposed to the delta function which has zero width). For what values of Δz is the delta function result roughly correct? [1 mark]

[TOTAL 15 marks]

4. Consider light reflecting from a metal surface at $z = 0$ where $z < 0$ is vacuum and $z > 0$ is metal. The incident wave (in complex notation) is,

$$\mathbf{E_i} = E_0 \exp i[k(z - ct)]\mathbf{x}$$

where $\mathbf{x}$ is the unit vector in the x direction.

(i) In the vacuum the waves obey the one dimensional Maxwell's equations in the form:

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z}$$

and

$$\frac{\partial E_x}{\partial t} = -c^2 \frac{\partial B_y}{\partial z}.$$

Find B_y for the incident wave. [2 marks]

(ii) Write down the form of E_x and B_y for the reflected wave. [2 marks]

(iii) Inside the metal $\mathbf{J} = \sigma \mathbf{E}$ where σ is the conductivity. The conductivity is high enough that the displacement current can be neglected. Thus

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z}$$

and

$$\mu_0 J_x = \mu_0 \sigma E_x = -\frac{\partial B_y}{\partial z}.$$

find a (complex) solution for E_x and B_y in the metal that is finite for $z \to \infty$. [6 marks]

(iv) Show that E_x and B_y must be continuous at $z = 0$. [2 marks]

(v) Show that in the limit of large conductivity where $\sqrt{\mu_0 \sigma c^2/\omega} \gg 1$ the reflected wave has the same intensity as the incident wave and the electric field in the conductor is negligible.

[3 marks]

[TOTAL 15 marks]

5. Consider waves propagating through the ionosphere. Let,

$$\mathbf{E_i} = E_0 \cos\left[kz - \omega t\right]\mathbf{x}$$

where **x** is the unit vector in the x direction.

(i) Show that electrons in this electric field have an oscillating velocity

$$\mathbf{v} = \frac{1}{\omega}\frac{e}{m_e}E_0 \sin\left(kz - \omega t\right)\mathbf{x}.$$

You may ignore the force on the electron due to the magnetic field of the wave and any z motion of the electrons.

[3 marks]

(ii) Using $\mathbf{J} = -ne\mathbf{v}$ (where n is the number density of electrons) and the one-dimensional Maxwell's equations,

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z}$$

and

$$\frac{1}{c^2}\frac{\partial E_x}{\partial t} = -\frac{\partial B_y}{\partial z} - \mu_0 J_x(z,t),$$

show that

$$\frac{\partial^2 E_x}{\partial t^2} = c^2\frac{\partial^2 E_x}{\partial z^2} - \omega_p^2 E_x$$

and find the constants c^2 and ω_p^2.

[4 marks]

(iii) Show that $\omega^2 = k^2c^2 + \omega_p^2$.

[2 marks]

(iv) Show that the phase velocity is always greater than c and that the group velocity is less than c.

[4 marks]

(v) Let z measure height from the ground. Radio waves are sent vertically up towards the bottom of the ionosphere at $z = z_0$. In the atmosphere $\omega_p = 0$ and in the ionosphere $\omega_p = \omega_0$ a constant. What happens to waves with $\omega < \omega_0$ when they encounter the bottom of the ionosphere?

[2 marks]

[TOTAL 15 marks]

6. In this question we consider waves propagating between two perfectly conducting parallel plates at $y = 0$ and $y = b$.

(i) Show that between the plates the electric field obeys the wave equation,

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}$$

you may need the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$. [3 marks]

(ii) Show that at the perfectly conducting plates $E_x = E_z = 0$. What about E_y and $\mathbf{B}$ at the conducting plates? [3 marks]

(iii) Show that

$$\mathbf{E} = E_0 \sin\left(\frac{n\pi y}{b}\right) \exp i[kz - \omega t]\mathbf{x}$$

is a solution of Maxwell's equations in the gap and find a formula for ω in terms of k. [5 marks]

(iv) Find the $\mathbf{B}$ field for the solution of part (iii). [2 marks]

(v) What happens to waves with $\omega^2 < \dfrac{n^2\pi^2}{b^2} c^2$. [2 marks]

[TOTAL 15 marks]

SECTION C

7. What is a Fabry-Perot interferometer? Illustrate with a diagram the physics of its operation and describe qualitatively why at normal incidence it transmits light at a set of equally spaced frequencies.

[5 marks]

(a) Show that the fraction I/I_0 of incident energy of vacuum wavelength λ transmitted at normal incidence for a plate spacing l is

$$I \propto \frac{1}{1 + \frac{4}{\pi} F^2 \sin^2(\delta/2)}$$

where $\delta = 4\pi n l/\lambda$, and n is the refractive index of the medium between the plates. Give expressions for F in terms of the reflectivity R and transmissivity T of the individual plates.

[4 marks]

(b) Find an expression for the resolution of the interferometer at any angle of incidence and draw a sketch of I as a function of the angle at the screen, indicating the width and separation of transmission maxima.

[4 marks]

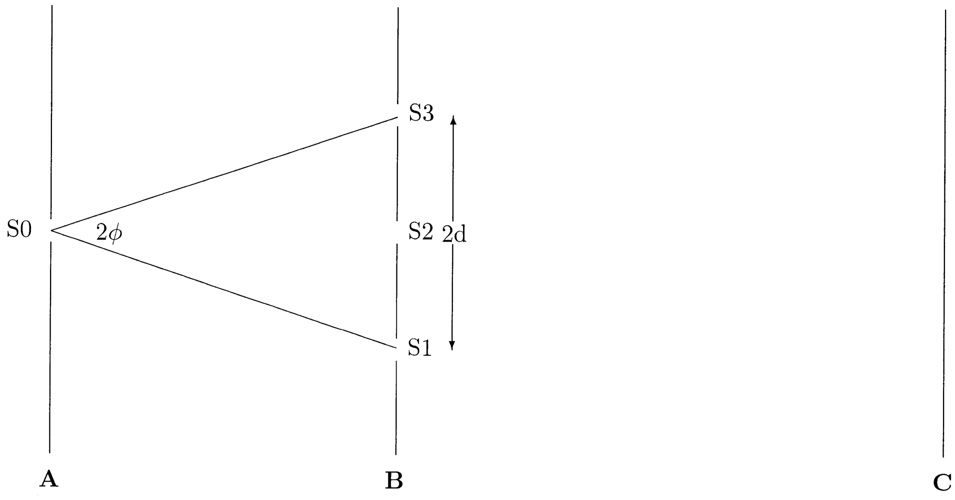
Compute the free spectral range for the wavelength $\lambda = 587.6$nm given that $l = 3.2$mm.

[2 marks]

[TOTAL 15 marks]

8. Describe Young's double slit arrangement for producing fringes. Draw the intensity pattern produced by the slits, labelling the values of the maxima and minima on your diagram.

[5 marks]



In the arrangement shown in the figure the screen **A** has a single slit and the screen **B** has three identical equally spaced slits. The length $2d$ from slit $S1$ to $S3$ subtends a small angle 2ϕ at the source slit $S0$. The screen **C** is placed far away so that you can assume the Fraunhofer limit.

Given that all 4 slits are assumed to be infinitesimally narrow show that

(a) The interference minima are all equal in intensity;

(b) The maxima alternate in intensity and compute the ratio of their intensities to that of the minima.

[10 marks]

[TOTAL 15 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

MATHEMATICS & STATISTICS OF MEASUREMENT

For Second-Year Physics Students

Thursday 29th May 2003: 14.00 to 16.00

Answer ALL parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0.$$

Use the Wronskian determinant to verify that the two functions in your solution are linearly independent.

[4 marks]

(ii) (a) Show that the Legendre Polynomial $P_2(x) = \frac{1}{2}(3x^2 - 1)$ satisfies the general relationship

$$\int_{-1}^{1} (P_n(x))^2 \, dx = \frac{2}{2n+1}.$$

[2 marks]

(b) Without calculation state what would happen if $P_n(x)P_m(x)$ instead of $(P_n(x))^2$ was used in this integral.

[1 mark]

(iii) Evaluate the following integrals in which $\delta(x)$ is the Dirac δ-function:

(a) $\int_{-\infty}^{\infty} \delta(x+2)\,(x+3)(x-2)\,dx$

(b) $\int_{-\infty}^{\infty} \delta(x)\,(27 - \cos x)\,dx$

(c) $\int_{-\infty}^{\infty} \delta(x - x')\,(4x' \sin x' - 2)\,dx'$

[3 marks]

[TOTAL 10 marks]

2. For this question you may, if you wish, use the statistics mode of the calculator provided.

[To use the statistics mode:
After switching the calculator on, press the ***MODE*** button (top row), followed by the number ***2*** key. The calculator is now in statistics mode. Enter data using the ***M+*** key. Use ***SHIFT*** plus the appropriate key to obtain the quantities you require. To return to normal mode, press ***MODE*** followed by the number ***1*** key.]

(i) A random variable can take the values 0, 1, 2, 3, 4 with probabilities 1/9, 1/6, 1/2, 1/6, 1/18 respectively. Calculate the mean value and standard deviation which would be expected from a large number of measurements of this variable.

[4 marks]

(ii) The Poisson distribution can be represented by the equation

$$P(\mu, k) = \frac{\mu^k e^{-\mu}}{k!}.$$

(a) State clearly what $P(\mu, k)$ represents, defining the parameters μ and k as part of your answer.

(b) A radioactive source is found to emit the following numbers of alpha particles in a number of experiments, each conducted over a 100 second period:

67 55 43 62 77 56 64 59 50 72

Calculate the average number of alpha particles emitted in 100 seconds and the standard error in this average count rate.

(c) What is the probability of the source emitting 5 alpha particles in a single shorter 10 second experiment?

[6 marks]

[TOTAL 10 marks]

SECTION B

3. (i) Show that, for integer n,

$$\int_0^{\frac{\pi}{2}} x \cos[(2n-1)x]\,dx = \frac{\pi}{2}\frac{(-1)^{n+1}}{(2n-1)} - \frac{1}{(2n-1)^2}\,.$$

[6 marks]

(ii) Show that the finite separable solutions (eigenfunctions) of the heat diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\,,$$

subject to the boundary conditions

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0 \text{ and } u\left(\frac{\pi}{2}, t\right) = 0$$

are

$$u_n(x,t) = A_n \cos[(2n-1)x]e^{-(2n-1)^2 t}\,.$$

[8 marks]

(iii) Given that the set of functions $u_n(x,t) = \cos[(2n-1)x]e^{-(2n-1)^2 t}$ are orthogonal over the range $0 \le x \le \frac{\pi}{2}$ and that $\int_0^{\frac{\pi}{2}} (u_n(x))^2\,dx = \frac{\pi}{4}$,

construct a linear combination, $\sum_{n=1}^{\infty} a_n u_n(x)$, of these eigenfunctions to satisfy the initial condition that $u(x,0) = f(x) = x$ for $0 \le x \le \frac{\pi}{2}$.

You may assume that

$$a_n = \frac{\int_0^{\frac{\pi}{2}} f(x)u_n(x)\,dx}{\int_0^{\frac{\pi}{2}} (u_n(x))^2\,dx}\,.$$

[4 marks]

(iv) Hence write down an expression for $u(x,t)$ for $t \ge 0$. [2 marks]

[TOTAL 20 marks]

4. In spherical polar coordinates Laplace's equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0\,.$$

(i) Assuming that there is no ϕ dependence in u, use the method of separation of variables with trial solution $u(r,\theta)=R(r)\Theta(\theta)$ and the separation constant $\ell(\ell+1)$, $(\ell=1,2,3\cdots)$, to show that the general radial part of the solution is

$$R_\ell(r)=Ar^\ell+\frac{B}{r^{\ell+1}}\,.$$

[10 marks]

(ii) Given that the solutions of

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\ell(\ell+1)\Theta=0$$

are the Legendre Polynomials $P_\ell(\cos\theta)$, write down an expression for $u_\ell(r,\theta)$. [2 marks]

(iii) Find the finite solution of Laplace's equation *inside* the sphere $r=3$, given that $u=1-\cos\theta$ on the surface of the sphere. [5 marks]

[The first few Legendre polynomials are:
$P_0(x)=1$; $P_1(x)=x$; $P_2(x)=\frac{1}{2}(3x^2-1)$; $P_3(x)=\frac{1}{2}(5x^3-3x)$.]

(iv) The orthogonality relationship for Legendre polynomials is

$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\sin(\theta)\,d\theta \quad = \quad \frac{2}{2n+1} \text{ for } m\neq n.$$

Identify the weight function and explain briefly how this orthogonality condition can be used to extend the results obtained above to obtain a solution of Laplace's equation *inside* the sphere $r=3$, given that on the surface the specified solution is $u(3,\theta)=f(\theta)$, i.e. write $u(r,\theta)$ as a series in $P_\ell(\cos\theta)$ and find an expression for the coefficients.

[3 marks]

[TOTAL 20 marks]

5. The Fourier transform $F(u)$ of the function $f(x)$ is given by

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux}\, dx .$$

(i) Verify the Fourier transform relationship

$$A \operatorname{rect}\left(\frac{x}{a}\right) \rightleftharpoons Aa \left(\frac{\sin \pi ua}{\pi ua}\right)$$

where $A \operatorname{rect}\left(\frac{x}{a}\right) = A$ for $|x| < \frac{a}{2}$ and is zero elsewhere. [4 marks]

(ii) Given that $f(x) \rightleftharpoons F(u)$ verify the shift theorem

$$f(x - x_0) \rightleftharpoons e^{-2\pi i x_0 u} F(u) .$$

[3 marks]

(iii) Given that $\delta(x) \rightleftharpoons 1$ evaluate $G(u)$, the Fourier transform of the function $g(x)$ which is defined by

$$g(x) = \delta(x + b) + 3\delta(x) + \delta(x - b) .$$

[3 marks]

(iv) Sketch $g(x)$ and evaluate $c(x)$ which is the convolution of rect $\left(\frac{x}{a}\right)$ with $g(x)$. [5 marks]

(v) Sketch $c(x)$ assuming that $b > a$. [2 marks]

(vi) Use the convolution theorem to show that the Fourier transform of $c(x)$ is given by

$$c(x) \rightleftharpoons a(3 + 2\cos(2\pi bu)) \left(\frac{\sin \pi ua}{\pi ua}\right) .$$

[3 marks]

[TOTAL 20 marks]

6. For this question you may, if you wish, use the statistics mode of the calculator provided.

[To use the statistics mode:
After switching the calculator on, press the ***MODE*** button (top row), followed by the number ***2*** key. The calculator is now in statistics mode. Enter data using the ***M+*** key. Use ***SHIFT*** plus the appropriate key to obtain the quantities you require. To return to normal mode, press ***MODE*** followed by the number ***1*** key.]

(i) A person shoots arrows at a circular target of outer radius a. It is found that the probability of an arrow hitting the target at a particular radius r is described by the continuous probability distribution $P(r)$ where

$$P(r)dr = C\left[1 - \left(\frac{r}{a}\right)^2\right] dr$$

and C is a constant.

(a) State clearly what probability is described by the quantity $P(r)dr$.

(b) Assuming that the target is always hit, find the value of the constant C.

(c) Under the same assumption, find the probability of hitting a centre circle of radius b on the target. Give your answer in terms of a and b. [7 marks]

(ii) (a) If $f(x, y, z) = x + y + z$ where x, y and z are three independent variables, derive an expression for the error s_f in f in terms of s_x, s_y and s_z, the errors in x, y and z respectively.

(b) A quantity x is measured n times, with the i^{th} measurement being denoted by x_i. Write down the expressions you would use to find the sample mean $\bar{x}$ and the sample standard deviation s.

(c) Show that the standard error in the sample mean is given by $s_{\bar{x}} = \dfrac{s}{\sqrt{n}}$. [8 marks]

(iii) An experiment to measure g, the acceleration due to gravity, yields the following results:

$$9.90 \quad 9.70 \quad 9.70 \quad 9.80 \quad 10.10 \quad 9.95 \quad 9.75 \quad 9.85\,\mathrm{m\,s^{-2}}.$$

(a) Compute the sample mean and standard error for this data set, quoting your results to an appropriate precision.

(b) Would you reject any of these points as unlikely to arise from the corresponding parent distribution? State the criteria you used to make your decision. [5 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

QUANTUM MECHANICS

For Second-Year Physics Students

Monday 9th June 2003: 10.00 to 12.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. The Hamiltonian corresponding to the magnetic interaction of a spin $\frac{1}{2}$ particle with charge e and mass m in a magnetic field $\mathbf{B}$ is

$$\hat{H} = -\frac{e}{m}\mathbf{B} \cdot \hat{\mathbf{S}}$$

where $\hat{\mathbf{S}}$ is the spin angular momentum operator (a suitable form is given below).

(i) Write down the energy eigenvalue equation for this particle in a field directed along the y-axis. [3 marks]

(ii) Calculate the energy eigenvalues and determine the corresponding normalised eigenvectors. [5 marks]

(iii) A particle, described at $t = 0$ by the spin state $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, interacts via the above Hamiltonian with the field B_y. Calculate the expectation value of S_z at $t = 0$. [2 marks]

(iv) Express the state vector χ as a superposition of energy eigenstates and hence write down the state vector $\chi(t)$ at a subsequent time t. [4 marks]

(v) Show that the expectation value of S_z executes simple harmonic motion with angular frequency

$$\omega = \frac{eB}{m} \, .$$

[6 marks]

[TOTAL 20 marks]

$$\left[\hat{\mathbf{S}} = \frac{\hbar}{2}\sigma : \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

2. A particle of mass m is bound in a central potential $V(r)$. Energy eigenfunctions have the form

$$u_{nlm}(\mathbf{r}) = R_{nl}(r) \cdot Y_l^m(\theta, \phi) \ .$$

The radial dependence is given by solutions to the equation

$$\left[\frac{-\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{l(l+1)}{2mr^2} + V(r) \right] \chi(r) = E \chi(r)$$

where $\chi(r) = r R_{nl}(r)$ and E is the energy.

(i) For a spherically symmetric infinite square well

$$\begin{aligned} V(r) &= 0 \qquad && r < a \\ &= \infty && r > a \end{aligned}$$

write down the boundary conditions on $R(r)$ at $r = 0$ and $r = a$.
Hence write down the boundary conditions for $\chi(r)$. [3 marks]

(ii) Obtain an expression for the energy eigenvalues of the s-states and show that the corresponding eigenfunctions are given by

$$u_{n00}(\mathbf{r}) = \frac{A}{r} \sin kr$$

where $k = n\pi/a$ and A is a normalization constant. [7 marks]

(iii) For a particle in the ground state calculate its average distance from the origin. [7 marks]

(iv) The above potential $V(r)$ is used to model the deuteron. This nucleus is formed by two nucleons of similar mass m, bound by their mutual attraction and separated by a distance r. Indicate if and how the energy level spectrum and the average separation differ from calculations in (ii) and (iii) above.

[3 marks]

[TOTAL 20 marks]

Integrals you may find useful:

$$\begin{aligned} \int \sin^2 kx \, dx &= \frac{x}{2} - \frac{\sin 2kx}{4k} \\ \int x \sin^2 kx \, dx &= \frac{x^2}{4} - \frac{x \sin 2kx}{4k} - \frac{\cos 2kx}{8k^2} \end{aligned}$$

3. Verify that position and momentum operators satisfy the commutation relation

$$[\hat{x}, \hat{p}_x] = i\hbar$$

and that operators (given below) for the spin angular momentum components of a spin $\frac{1}{2}$ particle satisfy

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z \ .$$

[5 marks]

Using these results show that the relation

$$(\Delta A)^2(\Delta B)^2 \geq \left(\left\langle\frac{i}{2}\left[\hat{A}, \hat{B}\right]\right\rangle\right)^2 \tag{3.1}$$

where ΔA and ΔB are the rms uncertainties in measurements of A and B, holds for the following (normalised) quantum mechanical states.

(i)

$$\psi = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp(-\frac{1}{2}\alpha x^2) \quad \text{where } \hat{A} = \hat{x} \text{ and } \hat{B} = \hat{p}_x \ .$$

[6 marks]

(ii)

$$\chi = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{where } \hat{A} = \hat{S}_x \text{ and } \hat{B} = \hat{S}_y \ .$$

[6 marks]

Comment on the physical significance of the above relation (equation 3.1).

[3 marks]

[TOTAL 20 marks]

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\sigma : \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\int_{-\infty}^{\infty} \exp(-ax^2)\, dx = \left(\frac{\pi}{a}\right)^{1/2} : \qquad \int_{-\infty}^{\infty} x^2 \exp(-ax^2)\, dx = \frac{1}{2a}\left(\frac{\pi}{a}\right)^{1/2} \ .$$

4. (i) Determine the momentum eigenfunction for a particle of momentum p, free to move along the x-axis. [3 marks]

(ii) A particle that is free to move along the x-axis is described by a wave function

$$\begin{aligned} \psi(x) &= a^{-1/2} \quad \text{in the range} \quad |x| \leq \frac{a}{2} \\ &= 0 \qquad\qquad\qquad\qquad |x| > \frac{a}{2} \; . \end{aligned}$$

(a) Show that the wave function in the momentum representation can be written

$$\phi(p) = A \, \frac{\sin q}{q} \qquad \text{where} \qquad q = \left(\frac{pa}{2\hbar}\right)$$

and A is a normalisation constant. [7 marks]

(b) Show that the probabilities of measuring momenta $0, \hbar/2a$ and $\hbar/a$ are in the ratio $1 : 4/\pi^2 : 0$. [2 marks]

(c) Sketch the momentum probability distribution and comment on its relation to the spatial probability distribution $|\psi(x)|^2$ and to the Heisenberg uncertainty principle. [3 marks]

(iii) A laser emits light of wavelength 500 nm. The transverse dimension of the beam at the laser is defined by an exit aperture of 1 cm. The beam is aimed precisely at a target 1 km distant. Use results obtained above to calculate the distance from the target centre of the first diffraction minimum. Estimate also the FWHM (full width at half maximum) spread of the beam at the target. [5 marks]

[TOTAL 20 marks]

5. A particle of mass m is confined by a one dimensional harmonic oscillator potential corresponding to a classical angular frequency ω.

(i) Write down the quantum mechanical Hamiltonian in terms of $\hat{x}$ and $\hat{p}$, the operators corresponding to position and momentum.
Show that it can be written in the form

$$\hat{H} = \hbar\omega\left(\alpha^2\hat{x}^2 + \beta^2\hat{p}^2\right)$$

where $\alpha^2 = m\omega/2\hbar$ and $\beta^2 = (2m\omega\hbar)^{-1}$. [3 marks]

(ii) By defining new operators

$$\begin{aligned} \hat{a} &= \alpha\hat{x} + i\beta\hat{p} \\ \hat{a}^\dagger &= \alpha\hat{x} - i\beta\hat{p} \end{aligned}$$

the energy eigenvalue equation for the oscillator may be written as

$$\left(\hat{a}\hat{a}^\dagger - \frac{1}{2}\right)u = \frac{E}{\hbar\omega}u$$

or as

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)u = \frac{E}{\hbar\omega}u\,.$$

Prove *one* of these two forms of the eigenvalue equation. [4 marks]

(iii) Show that the action of $\hat{a}$ on u is to convert it to another eigenstate $(\hat{a}u)$ of energy lower than that of u by an amount $\hbar\omega$. [4 marks]

(iv) The ground state u_0 has no eigenstates beneath it. By acting on u_0 with $\hat{a}$ determine the ground state energy eigenvalue E_0. [3 marks]

(v) Determine the (un-normalised) functional form of $u_0(x)$. [4 marks]

(vi) Use the raising operator $\hat{a}^\dagger$ to determine the (un-normalised) functional form of $u_1(x)$. [2 marks]

[TOTAL 20 marks]

6. Write *short* notes on FOUR of the following topics:

(i) The time dependence of the wave function.

(ii) Compatible and complementary observables.

(iii) The main features of the Stern-Gerlach apparatus and the evidence it provides for the spin of the electron.

(iv) Conserved quantities in quantum mechanical systems.

(v) α-decay of radioactive nuclei.

(vi) Compton scattering and the evidence it provides for the particle-like properties of X-rays.

(vii) "A philosopher once said that it is necessary for the very existence of science that the same initial conditions always produce the same results. *Well they don't!* "
R.P.Feynman (extract from *The Character of Physical Law*).

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ELECTRONS IN SOLIDS and APPLICATIONS OF QUANTUM MECHANICS

For Second-Year Physics Students

Thursday 12th June 2003: 10.00 to 12.00

Answer ALL parts of Section A, ONE question from Section B and ONE question from Section C.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

SECTION A

1. (i) A pair of simple un-normalised trial wavefunctions for the H_2^+ molecule can be constructed as

$$\begin{aligned}\Psi_g &= \psi_{1s}(r_A) + \psi_{1s}(r_B)\\ \Psi_u &= \psi_{1s}(r_A) - \psi_{1s}(r_B)\end{aligned}$$

What is the origin of the wavefunctions $\psi_{1s}(r_A)$, $\psi_{1s}(r_B)$ used here? Sketch Ψ_g and Ψ_u as a function of the internuclear co-ordinate. Identify which of these wavefunctions leads to bonding and say why.

[4 marks]

(ii) Write down an expression for the rotational energy of a diatomic molecule. Show that the energy separation between two adjacent rotational levels is given by 2B(J+1) where J is the rotational quantum number of the lower level and B is the rotational constant of the molecule.

[3 marks]

[TOTAL 7 marks]

2. The energy-wavevector dispersion curve for a hypothetical one-dimensional metal has the following form:

$$E = 2A \sin^2(\tfrac{1}{2}ka) ,$$

where a is the crystal lattice spacing and A is a constant.

(i) If the effective mass m^* is equal to the free electron mass m for small wavevector, determine the constant A.

[3 marks]

(ii) Determine dE/dk at the Brillouin zone boundaries.

[3 marks]

(iii) Derive an expression for the effective mass m^*.

[3 marks]

(iv) Given that the density of states per unit length between wavevectors k and $k+dk$ is $(1/\pi)\,dk$, as for a free electron system, show that the density of states between energies E and $E+dE$ is $ma/\pi\hbar^2$ at $k=\pi/2a$ and becomes infinite at $k=\pi/a$.

[4 marks]

[TOTAL 13 marks]

SECTION B

3. (i) What assumptions are made in the free electron theory of metals? In what respect does this theory fail to account for other types of material?

[3 marks]

(ii) The quantum mechanical energy levels E_n of an electron of mass m in a one-dimensional box of length L is

$$E_n = \frac{h^2 n^2}{8mL^2} ,$$

where n is a positive integer. Consider a collection of 8 "atoms", modelled as one-dimensional boxes of length a. If each atom has one valence electron, show that the energy of the electrons in the 8 atoms is

$$E_1 = \frac{h^2}{ma^2} .$$

The atoms are brought together to form a system of length $8a$ over which the electrons move freely. Show that the energy is now given by

$$E_2 = \frac{15h^2}{128ma^2} .$$

Discuss this result in the context of the free electron theory of metals.

[6 marks]

(iii) Each quantum state occupies a volume $\Delta\Omega = 4\pi^3/V$ in k-space, where V is the volume of the metal. Show that the number dN of such quantum states with values of k in the range k and $k + dk$ is

$$dN = \frac{Vk^2}{\pi^2}\, dk .$$

Then, using the E–k dispersion relation for free electrons, show that the density states is

$$D(E) = \frac{dN}{dE} = \frac{V}{2\pi^2\hbar^3}(2m)^{3/2}E^{1/2} .$$

[6 marks]

(iv) Determine an expression for the Fermi energy of a free electron metal with $n = N/V$ electrons per unit volume. Explain the significance of the Fermi energy in determining the properties of such a metal.

[5 marks]

[TOTAL 20 marks]

4. Explain the difference between metals, insulators, and semiconductors in terms of the filling of electronic bands. Include in your explanation the ease with which electrons can be thermally excited from occupied to empty states.

[5 marks]

The density of free electrons n and holes p in a semiconductor can be written as

$$n = N_c \, \mathrm{e}^{-(E_g-\mu)/k_\mathrm{B}T}, \qquad p = N_v \, \mathrm{e}^{-\mu/k_\mathrm{B}T} ,$$

where the zero of energy has been taken at the top of the valence band and all symbols have their usual meanings.

(i) Explain the difference between an intrinsic and an extrinsic semiconductor. Show that, for an intrinsic semiconductor,

$$p = n = \sqrt{N_c N_v} \, \mathrm{e}^{-E_g/2k_B T} .$$

[5 marks]

(ii) Given that $N_c \propto m_c^{*3/2}$ and $N_v \propto m_v^{*3/2}$, with the same constant of proportionality, show that

$$\mu = \tfrac{1}{2} E_g + \tfrac{3}{4} k_B T \ln\left(\frac{m_v^*}{m_c^*}\right) .$$

What is the value of the chemical potential μ at $T = 0$? Under which circumstances is this the *exact* value of the chemical potential at *all* temperatures?

[5 marks]

(iii) Explain in physical terms the behaviour of the chemical potential if $m_v^* > m_c^*$ and if $m_c^* > m_v^*$.

[5 marks]

[TOTAL 20 marks]

SECTION C

5. (i) List any three deficiencies of the Bohr model for hydrogen. [3 marks]

(ii) Any region of space for which the kinetic energy would become negative is classically forbidden. Solution of the Schrödinger equation shows that the energy levels of hydrogen are given by

$$E = -\frac{1}{2n^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m}{\hbar^2},$$

where the symbols take their usual meanings. Show that the classically forbidden region for an electron in the ground state of hydrogen is given by $r > 2a_0$ where $a_0 = 4\pi\epsilon_0\hbar^2/me^2$. [4 marks]

(iii) Show that the probability P_{cf} of finding a ground state electron in hydrogen in a classically forbidden region of space is $13e^{-4}$ (~ 0.238). For hydrogen-like systems

$$\psi_{1s} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}\,.$$

You may require the standard integral

$$\int x^2 e^{-x} dx = -e^{-x}(x^2 + 2x + 2)\,.$$

[10 marks]

(iv) What is the classically forbidden region for the 2s state? Without performing any calculations, would you expect P_{cf} for the 2s state to be larger or smaller than P_{cf} for the 1s state? [3 marks]

[TOTAL 20 marks]

6. (i) Write down the Schrödinger equation for He and describe the physical meaning of each term. What is the independent particle approximation (IPA)? Show that, according to the IPA, the first ionisation threshold of He would be at 54.4eV. [10 marks]

(ii) Table below lists the lowest energy levels in He as found by experiment

Configuration	Term	Energy (eV)
$1s^2$	^3S	—
	^1S	-24.58
1s2s	^3S	-4.77
	^1S	-3.97
1s2p	^3P	-3.62
	^1P	-3.37

(a) Comment on the discrepancy between the experimental value for the ground state energy and the value given by the IPA. [2 marks]

(b) Explain why there is no entry in the table for the $1s^2\ {}^3S$ level? [6 marks]

(c) Why is the splitting between the two 1s2s terms greater than that between the two 1s2p terms? [2 marks]

[TOTAL 20 marks]

The ionisation potential of H is 13.6eV.

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

THERMODYNAMICS & STATISTICAL PHYSICS

For Second-Year Physics Students

Tuesday 10th June 2003: 10.00 to 12.00

Answer ALL parts of Section A, ONE question from Section B and ONE question from Section C.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) The state of a gas is characterized by its pressure P, temperature T, volume V, internal energy U and entropy S. State, without proof, whether each of these variables is intensive or extensive. State whether each of the combinations PV and TS is intensive or extensive.

[2 marks]

(ii) The Gibbs free energy is defined by $G(T, P, N) = U - TS + PV$ where N is the number of particles, and we assume that there is only one type of particle present. Explain why the N-dependence of G is given by

$$G(T, P, N) = Ng(T, P)$$

where $g(T, P)$ is independent of N.

[2 marks]

(iii) Consider an ideal gas of N particles at temperature T. Write down the number of degrees of freedom and the expression for the internal energy in the case where the gas is (a) monatomic, and (b) diatomic. Show that your answer in the diatomic case may be explained in terms of the monatomic case by considering a monatomic gas of $2N$ particles with the particles connected pairwise by rigid rods.

[4 marks]

(iv) Write down the equation describing a quasistatic adiabatic expansion of an ideal gas, in terms of P and V, and the gas constant γ, which you should define.

Show that the work done during a quasistatic adiabatic expansion of an ideal gas, from pressure P_1 and volume V_1 to pressure P_2 and volume V_2 is

$$W = \frac{1}{\gamma - 1}\left[P_1 V_1 - P_2 V_2\right].$$

Use the equation of state of an ideal gas to show that the answer may be written,

$$W = c_V (T_2 - T_1)$$

where T_1 and T_2 are the initial and final temperatures and c_V is the specific heat at constant volume.

Which gives the greater amount of work for the same change: a monatomic or diatomic gas (of the same number of particles)? Give a brief physical explanation of your answer.

[7 marks]

[TOTAL 15 marks]

2. (i) The Fermi-Dirac distribution function is

$$f(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon-\mu}{kT}\right)+1}$$

where ϵ is the energy, μ the chemical potential (Fermi energy) and T is temperature. Without doing any calculation, draw a picture of this distribution as a function of energy ϵ, in the limit of zero temperature, clearly marking the value $\epsilon = \mu$. [2 marks]

(ii) Give a brief physical explanation for the form of this distribution at zero temperature. [2 marks]

(iii) Name two physical systems for which the Fermi-Dirac distribution is important. [1 mark]

[TOTAL 5 marks]

SECTION B

3. (i) A heat engine E operates in a cycle between reservoirs R_{in} and R_{out} at temperatures, T_{in} and T_{out} respectively, where $T_{out} < T_{in}$. It draws in heat Q_{in} from R_{in}, deposits heat Q_{out} in R_{out} and produces work $W = Q_{in} - Q_{out}$. Draw a diagram depicting this engine. Define the efficiency η of the engine in terms of W and Q_{in}.

Without doing any calculation, write down the efficiency of the engine in terms of T_{in} and T_{out}, in the case where the engine is a Carnot engine. By equating with your first expression for the efficiency, write down an expression for the work W produced by a Carnot engine in terms of Q_{in}, T_{out} and T_{in}.

[4 marks]

(ii) State the two forms of the Second Law of thermodynamics: (a) in terms of the restrictions on converting heat into work (the Kelvin-Planck statement), and (b) in terms of the restrictions on the transfer of heat from a cooler to a hotter body (the Clausius statement). In each case draw a diagram depicting an engine which violates each statement of the second law.

[4 marks]

(iii) By combining engines of the types drawn in parts (i) and (ii), show that an engine which violates the Clausius statement of the second law, (b), must violate the Kelvin-Planck statement, (a).

[3 marks]

(iv) Suppose that the reservoir R_{in} at temperature T_{in} is connected via a lagged bar to a cooler reservoir R'_{in} at temperature $T'_{in} < T_{in}$ and heat Q_{in} flows from R_{in} to R'_{in}. Is this process reversible or irreversible? Compute the total entropy change ΔS of the two reservoirs.

[3 marks]

(v) Now suppose that the cooler reservoir R'_{in} in part (iv) is used to run the Carnot engine in part (i) (where $T'_{in} > T_{out}$). Write down an expression for the amount of work W' produced, in terms of Q_{in}, T_{out} and T'_{in}. Is it greater or less than the work W produced with the engine described in part (i)? Derive a simple expression relating $|W - W'|$ to ΔS. Use this result to give an interpretation of the significance of the magnitude of ΔS.

[6 marks]

[TOTAL 20 marks]

4. (i) Write down the fundamental equation of thermodynamics, defining all of the quantities that appear.

[2 marks]

(ii) Write down the internal energy and the equation of state for an ideal gas with specific heat capacity at constant volume c_V. Use them, together with the fundamental equation, to show that the entropy of an ideal gas is,

$$S = c_V \ln\left(\frac{T}{T_0}\right) + Nk \ln\left(\frac{V}{V_0}\right) + S_0$$

where S_0 is a constant.

[4 marks]

(iii) Rewrite the fundamental equation in (i) in terms of the Helmholtz free energy $F = U - TS$, and hence show that F has T and V as dependent variables. Use the results of part (ii) to write down the Helmholtz free energy of an ideal gas.

[3 marks]

(iv) Consider a system which is initially in equilibrium at temperature T with a reservoir. Suppose the system is allowed to draw heat from the reservoir and produce an amount of work W. We suppose that the whole process takes place a constant temperature T. By considering the change in the total entropy of the system together with the reservoir, show that the maximum amount of work extractable from the system is equal to the decrease in the free energy, that is,

$$W \leq -\Delta F.$$

Would you expect this result to still hold if the temperature varies during the process, but the initial and final temperatures are the same?

[8 marks]

(v) Compute the work done by an ideal gas during a quasistatic isothermal expansion from volume V_1 to V_2 at temperature T. Use the results of parts (iii) and (iv) to show that this process in fact produces the maximum amount of extractable work.

[3 marks]

[TOTAL 20 marks]

SECTION C

5. Consider an isolated assembly of distinguishable, weakly interacting particles (eg atoms in a lattice) with non-degenerate energy states (levels) ε_j and occupation numbers n_j. The system is subject to the following constraints on total energy, U, and total particle number, N.

$$U = \sum_j n_j \varepsilon_j \tag{5.1}$$

and

$$N = \sum_j n_j \tag{5.2}$$

(i) Write down Boltzmann's definition of entropy, S, as a function of Ω, the number of different ways that the N particles may occupy the energy states. Explain the physical meaning of Boltzmann's formulation in terms of disorder. Support your explanation with an example of how this concept might apply to free adiabatic expansion.

[6 marks]

(ii) Write down the expression for Ω in terms of $N!$ and $n_j!$ for this system, and, using Stirling's approximation, $\ln N! \approx N \ln N - N$ for large N, show that S may be written as

$$\frac{S}{k_B} = N \ln N - \sum_j n_j(\alpha + \beta\varepsilon_j) \tag{5.3}$$

where $n_j = \exp(\alpha + \beta\varepsilon_j)$ and α and β are constants. State the definition of the partition function, Z, and show how α is determined in terms of Z. State the relationship between β and temperature.

[6 marks]

(iii) Consider a system comprising three energy levels with energy levels $\varepsilon_0 = 0$, $\varepsilon_1 = 1.38 \times 10^{-23}\,\mathrm{J}$ and $\varepsilon_2 = 2.76 \times 10^{-23}\,\mathrm{J}$. In this case let the energy levels be degenerate, with $g_0 = 2$, $g_1 = 2$ and $g_2 = 1$.

In general terms, how are these energy levels populated in the limiting cases of very low ($T \to 0$) and very high ($T \to \infty$) temperature?

Calculate values of Z for temperatures of 0.1, 1, 10, and $10^3\,\mathrm{K}$. Plot a graph of Z against $\log_{10} T$ using your results.

Show, using equation (5.3) above that in the high temperature limit the entropy is given by $S = k_B \ln 5^N$.

[8 marks]

[TOTAL 20 marks]

6. The Bose-Einstein distribution (the mean number of bosons in a single energy state) is given for the kth group of energy levels by

$$f_k = \frac{N_k}{g_k} = \frac{1}{\exp(-\alpha - \beta\varepsilon_k) - 1} \tag{6.1}$$

where α and β are constants, $\varepsilon_k =$ energy and N_k number of particles in $\mathrm{k^{th}}$ group of energy levels, $g_k =$ number of energy states in $\mathrm{k^{th}}$ group.

(i) Explain why, for the case of a photon gas in an enclosure, the number of photons per energy state reduces to

$$f(\nu) = \frac{1}{\exp\left(\dfrac{h\nu}{k_B T}\right) - 1}\,. \tag{6.2}$$

[7 marks]

(ii) Given that the density of states in k-space is given by

$$g(k)dk = \frac{4\pi k^2 V}{(2\pi)^3}dk \tag{6.3}$$

where $k = 2\pi/\lambda =$ wave vector $= 2\pi\nu/c$, and V = volume of the enclosure, convert this to the density of states in ν-space, $g(\nu)$ to show that the photon energy in the range ν to $d\nu$ is given by

$$u(\nu)d\nu = [(g(\nu)f(\nu)d\nu]h\nu = \frac{8\pi h\nu^3 V}{c^3}\,\frac{1}{\exp\left(\dfrac{h\nu}{k_B T}\right) - 1}d\nu \tag{6.4}$$

which is known as the Planck distribution. k_B and h are the usual Boltzmann and Planck constants, and c is the speed of light.

[7 marks]

(iii) Integrate equation (6.4) over all ν to obtain the total energy density:

$$\frac{u}{V} = \int_0^\infty \frac{u(\nu)}{V}d\nu \quad \text{using the standard integral} \int_0^\infty \frac{x^3dx}{e^x - 1} = \frac{\pi}{15} \quad \text{with } x = \frac{h\nu}{k_B T},$$

and show that the total energy density, u/V, depends on T^4. Use your results to calculate the value of the Stefan-Boltzmann constant, $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$, which expresses the energy per square metre emerging from a black cavity enclosure with energy density u/V inside it. This rate of radiation per unit area is related to u/V by $\sigma T^4 = \left(\frac{u}{V}\right)\frac{c}{4}$.

[6 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

SUN, STARS and PLANETS

For Second-Year Physics Students

Wednesday 28th May 2003: 10.00 to 12.00

Answer ALL parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Fundamental physical constants

a	radiation density constant	$7.55 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$
c	speed of light	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
e	magnitude of charge on electron	$1.60 \times 10^{-19}\,\mathrm{C}$
G	gravitational constant	$6.67 \times 10^{-11}\,\mathrm{N\,m^{2}\,kg^{-2}}$
h	Planck's constant	$6.62 \times 10^{-34}\,\mathrm{J\,s}$
k	Boltzmann's constant	$1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
m_e	mass of electron	$9.11 \times 10^{-31}\,\mathrm{kg}$
m_H	mass of hydrogen atom	$1.67 \times 10^{-27}\,\mathrm{kg}$
N_A	Avogadro's number	$6.02 \times 10^{23}\,\mathrm{mol^{-1}}$
σ	Stefan-Boltzmann constant	$5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\ K^{-4}}$
ε_0	permittivity of free space	8.85×10^{-12} farad $\mathrm{m^{-1}}$
μ_0	permeability of free space	$4\pi \times 10^{-7}$ henry $\mathrm{m^{-1}}$
$\mathcal{R}$	gas constant (k/m_H)	$8.26 \times 10^{3}\,\mathrm{J\,K^{-1}\,kg^{-1}}$
eV	electron-volt	$1.6 \times 10^{-19}\,\mathrm{J}$

Astronomical quantities

$L_\odot$	luminosity of Sun	$3.83 \times 10^{26}\,\mathrm{W}$
$M_\odot$	mass of Sun	$1.99 \times 10^{30}\,\mathrm{kg}$
$R_\odot$	radius of Sun	$6.96 \times 10^{8}\,\mathrm{m}$
$T_{\mathrm{eff}\odot}$	effective temperature of Sun	$5780\,\mathrm{K}$
SNU	solar neutrino unit	10^{-36} interactions $\mathrm{s^{-1}}$ target atom^{-1}
AU	astronomical unit	$1.50 \times 10^{11}\,\mathrm{m}$
pc	parsec	$3.09 \times 10^{16}\,\mathrm{m}$

SECTION A (Compulsory)

1. (i) In a plane-parallel atmosphere the pressure $p(z)$ and density $\rho(z)$ depend only on the coordinate z (upwards). Take the gravitational acceleration g to be constant. Derive from first principles an inequality that must hold (Schwarzschild's criterion) for the atmosphere to be stable to convection.

[5 marks]

(ii) A planet orbits at a distance D from a star whose luminosity is L and effective temperature is T_{eff}. The planet has mass M_p, radius R_p and albedo a. Assuming the greenhouse effect to be negligible, obtain an estimate of the planet's surface temperature T_p.
Use your answer to estimate the surface temperature of Mercury.

[Take Mercury's mass, radius and albedo to be 3.3×10^{23} kg, 2400 km and 0.11 respectively, and the Sun-Mercury distance to be 58×10^{6} km.]

[5 marks]

(iii) Radiation from the surface of a star may be approximated as that of a black body and so is given by Planck's law, $B_\lambda(T) = (2hc^2/\lambda^5)/[\exp(hc/\lambda kT)–1]$.

For a given effective surface temperature T, show that the wavelength of maximum emission λ_{max} is inversely proportional to T. For the Sun, $\lambda_{\text{max}} = 500$ nm and $T = 5800$ K. What is the value of λ_{max} for a star with surface temperature 3500 K? What colour would the star appear to be?

[4 marks]

(iv) Calculate the escape velocity from Venus's exosphere (about 200 km above Venus's surface).

[Take Venus's mass and radius to be 5×10^{24} kg and 6×10^{6} m respectively.]

[2 marks]

(v) Given the Sun's mass is 2×10^{30} kg and its radius is 7×10^{8} m, estimate the Sun's mean density and central pressure. Estimate also its central temperature. Comment on the accuracy of your estimates.

[5 marks]

(vi) Calculate using classical physics the distance of closest approach of two protons of energies 2 keV in a stellar core.

[3 marks]

[TOTAL 24 marks]

SECTION B

2. (i) The first two steps of the p-p nuclear reaction chain in the Sun are

$$
\begin{array}{ccccccccc}
{}^{1}H & + & {}^{1}H & \longrightarrow & \_\_\_ & + & \_\_\_ & + & \nu_e \\
\_\_\_ & + & \_\_\_ & \longrightarrow & {}^{3}\mathrm{He} & + & \gamma & &
\end{array}
$$

Write out the above equations correctly, filling in the four blanks. [4 marks]

(ii) Thereafter, the p-p reaction proceeds to follow one of three branches.
You are reminded that Branch I proceeds with

$$ {}^{3}\mathrm{He} + {}^{3}\mathrm{He} \longrightarrow {}^{4}\mathrm{He} + 2\,{}^{1}\mathrm{H} $$

while Branches II and III both initially proceed with

$$ {}^{3}\mathrm{He} + {}^{4}\mathrm{He} \longrightarrow {}^{7}\mathrm{Be} + \gamma $$

(the ^{4}He in the above reaction is merely a catalyst, its destruction leading to the creation of two ^{4}He rather than just one).

The effective energy release from Branch I is 26.2 MeV while from Branches II and III it is 25.7 MeV and 19.1 MeV respectively.

How many neutrinos from the primary p-p fusion reaction – i.e. the first reaction in part (i) – per ^{4}He created does each of Branch I, Branch II and Branch III create?

Assuming that the solar luminosity is due to p-p hydrogen burning, the expected luminosity of neutrinos from the primary p-p fusion can be estimated if the fraction of terminations in each branch is known. Obtain upper and lower bounds for the expected luminosity of such neutrinos. [7 marks]

(iii) The standard solar model predicts that 85% of p-p reactions in the Sun terminate in Branch I. Estimate the expected flux of primary p-p neutrinos. Assuming an unchanging electron-neutrino luminosity which is radiating isotropically from the centre of the Sun, what is the expected neutrino flux at Earth. (Here, flux means the number of particles passing through unit area in unit time.)

[3 marks]

(iv) Explain briefly what is the so-called solar neutrino problem, and what is the resolution of the problem. [4 marks]

[TOTAL 18 marks]

3. (i) Sirius A has an apparent visual magnitude $m_{\mathrm{V}} = -1.5$ and a spectral type A0V (which you may take corresponds to $T_{\mathrm{eff}} = 10,000\,\mathrm{K}$ and bolometric correction -0.4). Its distance from the Earth is $2.7\,\mathrm{pc}$. Calculate its luminosity and radius.

[Reddening by interstellar dust can be neglected. You may assume that a star with absolute bolometric magnitude $M_{\mathrm{bol}} = 0$ has a luminosity of $3 \times 10^{28}\,\mathrm{W}$.] [7 marks]

(ii) Sirius B is a binary companion to Sirius A. It has the same spectral type and its apparent visual magnitude is $m_{\mathrm{V}} = 8.5$. Calculate its luminosity and radius. [4 marks]

(iii) Measurements of the orbits of Sirius A and B give a binary period of 50 years, mass ratio $M_{\mathrm{A}}/M_{\mathrm{B}} = 3$ and a semi-major axis of 8 arc-seconds.
Calculate the masses and mean densities of both components. Comment on the nature of the two stars. [7 marks]

[TOTAL 18 marks]

4. (i) An extrasolar planet of mass m orbits a distant star of mass M ($M \gg m$). As seen from Earth, the orbital plane is at an angle i to the line of sight. Assuming circular orbits, derive expressions for the distance D between the planet and the star, and also for the line-of-sight orbital velocity of the star, in terms of m, M, i and the orbital period P. [8 marks]

(ii) The Doppler shift of a distant star of one solar mass is found to have a periodic component of amplitude $50\,\mathrm{m\,s^{-1}}$ and period 80 days. Assuming this is due to a planet orbiting edge-on as seen from Earth, calculate the distance D between the planet and the star, and the planet's mass m. Give your answers in units of the astronomical unit and the mass of Jupiter ($M_\mathrm{J} = 1.9 \times 10^{27}\,\mathrm{kg}$). How do these values for this planet compare with those of known extrasolar planets? [5 marks]

(iii) A transit-detection project observes the same star in white light in the hope of detecting the planet in (ii). Again assume the orbit is edge-on. If the star has the same radius as the Sun, the planet and star have similar mean densities, and the star has apparent visual magnitude $m_\mathrm{V} = 4.0$, what will be the change in apparent magnitude when the planet transits the stellar disk? [5 marks]

[TOTAL 18 marks]

5. Write concisely (1-2 pages each) on TWO of the topics below. In what you write, draw attention to the main features of the phenomena involved, and identify the relevant physics underpinning our understanding of them. Illustrate your answer with sketch diagrams where this is helpful.

(i) The Sun's coronal magnetic field.

(ii) The comparative histories of the atmospheres of Venus and Earth.

(iii) Jupiter's Galilean moons.

(iv) Kuiper Belt Objects.

(v) Planetary rings.

[TOTAL 18 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

PHYSICS I Comprehensive Paper

For Third - and Fourth -Year Physics Students

Friday 23rd May 2003: 10.00 to 13.00

Answer Five questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. You have been hired to build and manage a dam to prevent a river from overflowing. A natural spring guarantees that the level h of the river is always positive, i.e., $h \geq h_{min} > 0$. Measurements of the level of the river once a day show that it is described by a power-law probability density,

$$P(h) = \begin{cases} Ah^{-2} & \text{for } h \geq h_{min} \\ 0 & \text{otherwise,} \end{cases}$$

where A is a constant. You may assume that there are no correlations between the daily measured levels of the river.

(i) Show that the probability density is normalised if $A = h_{min}$. [2 marks]

(ii) (a) What is the probability, on a given day, of measuring a level $h \geq h_{max}$ with $h_{max} \geq h_{min}$? [3 marks]

(b) On average, how many days would you have to wait to see one event with $h \geq h_{max}$? [2 marks]

(iii) Your job contract has a clause that if the dam overflows, you get fired.

(a) Can you build a dam that would guarantee your job forever? Explain your answer. [3 marks]

(b) You would like to keep your job for N years with probability p. How high should you build the dam? [4 marks]

(c) Find the height of the dam, assuming you want to keep your job for 10 years with 90% probability when $h_{min} = 0.01\,\text{m}$. [1 marks]

(iv) (a) Find the average level $\langle h \rangle$ of the river. [2 marks]

(b) Suggest a modification of the probability density for $P(h)$ that would make the model more physically realistic. [3 marks]

[TOTAL 20 marks]

2. The colour of different red, orange and yellow fruits and vegetables is due to the absorption of light by an organic molecule called a carotenoid. This molecule is a long, thin ribbon-like molecule. It is made up of a chain of identical repeating chemical units. If the length of each chemical unit is d and there are N units, the total length of the molecule is $L = Nd$. Use the following steps to derive the relationship between N and the wavelength of absorbed light λ.

(i) Write down the time-independent Schrödinger equation in 1 dimension. [2 marks]

(ii) Assume that the molecule can be thought of as a one-dimensional box between $x = 0$ and $x = L$. Let the potential inside the molecule be $U(x) = 0$ and outside the molecule be infinite. Consider a single electron on the molecule. Derive (a) the wavefunction $\psi(x)$ in terms of wavevector k and (b) k as a function of L and n, the quantum number of the state. [5 marks]

(iii) Find the energy E of the electron as a function of n, N and d. [3 marks]

(iv) Assume that the number of electrons on each chemical unit is 2. In a molecule of N units, what is the quantum number of the highest filled state in terms of N? (Assume that the temperature is zero). [2 marks]

(v) The lowest energy transition of the molecule will result in the absorption of a photon of energy E_{abs}. Derive the relationship between E_{abs} and N. [4 marks]

(vi) Hence show that for $N \gg 1$:

$$\lambda \approx Ncd^2 \frac{4m_e}{h}.$$

The absorption of light in red tomatoes and yellow lemons begins at 618 and 515 nm respectively. If $d = 0.25$ nm, calculate the molecular length L of the carotenoid involved. [4 marks]

[TOTAL 20 marks]

3. (i) Write down an expression for the Lorentz force on a particle of charge $q > 0$ moving in an electric field $\mathbf{E}$ and a magnetic induction $\mathbf{B}$.

[2 marks]

A Penning ion trap is a device used for confining charged particles to a small region of space. The trap electrodes generate a quadrupole electrostatic potential given by

$$V(x, y, z) = A\left(2z^2 - x^2 - y^2\right)$$

(where A is a constant) and there is a superimposed uniform magnetic induction $\mathbf{B} = B\hat{\mathbf{z}}$.

(ii) (a) Write down an expression for the z-component of the total electromagnetic force acting on a moving particle of charge q in the trap and explain why it does not depend on B.

(b) Hence write an equation of motion for the z-component of the particle's motion and show that it consists of simple harmonic motion at an angular frequency $\omega_z = (4qA/m)^{1/2}$, where m is the mass of the particle.

[4 marks]

(iii) From your expression for the Lorentz force, show that the equation of motion for x leads to the following differential equation:

$$\ddot{x} = (q/m)(2Ax + B\dot{y})$$

and find the equivalent expression for $\ddot{y}$.

[4 marks]

(iv) These equations can be solved by casting them in a complex form with $r = x + iy$. Hence (or otherwise) show that one possible motion in the x, y plane consists of a circular orbit at an angular frequency

$$\omega_1 = \Omega/2 - \left(\Omega^2/4 - \omega_z^2/2\right)^{1/2}$$

or

$$\omega_2 = \Omega/2 + \left(\Omega^2/4 - \omega_z^2/2\right)^{1/2}$$

where $\Omega = q|B|/m$ is the angular cyclotron frequency of the ion. Hence show that the trap is only stable if $B^2 > 8Am/q$.

[6 marks]

(v) Magnesium has an atomic mass of 24. Find ω_1, ω_2 and ω_z for a singly charged positive magnesium ion in a trap with $B = 1$ T and $A = 2 \times 10^5\ \mathrm{V\,m^{-2}}$.

[4 marks]

[TOTAL 20 marks]

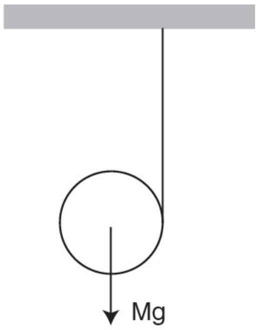
4. (i) In a rotating system with moment of inertia I, what is the relationship between angular acceleration and applied torque?

[2 marks]

(ii) A rotating body of mass M and moment of inertia I has angular velocity ω and its centre of mass has linear velocity v. What is its total kinetic energy?

[2 marks]

(iii) A simple yoyo, as shown in the figure, is constructed from a cylinder of mass M and radius R, whose moment of inertia about its axis is $I = MR^2/2$. A thin string of length $L(\gg R)$ is fixed to it and wrapped around the cylinder, which is dropped from rest and unwinds the string. Just before reaching the end of its fall, the axis of the cylinder is level with the end of the string. At this point:



(a) What fraction of the kinetic energy of the yoyo is rotational?

(b) Derive an expression for the angular velocity of the yoyo about its axis.

(c) Explain why the downward acceleration of the yoyo is constant during its fall.

(d) Use your answers to the previous parts to calculate the time taken for the yoyo to reach this position.

[9 marks]

(iv) After its fall, the yoyo begins to move up the string.

(a) Explain why, and find to what height the yoyo will rise.

(b) Suggest why a real yoyo is made with a thin axle around which to wind the string.

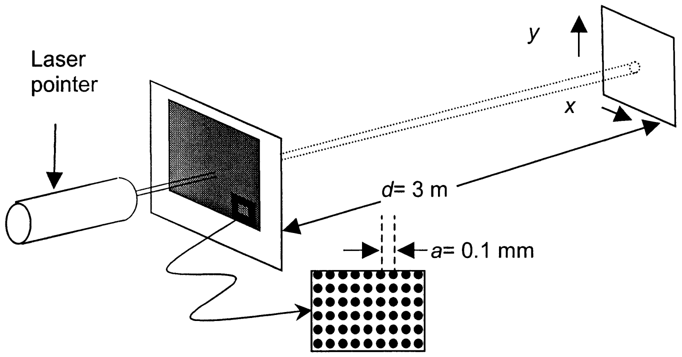
[4 marks]

(v) In a real yoyo, the string is not fixed to the reel but wound loosely around it in a loop. At the end of the yoyo's downward motion the player is advised to tug the string gently upwards. What is the reason for applying this force and what is likely to happen if the string is not pulled?

[3 marks]

[TOTAL 20 marks]

5. A laser diode pointer is fabricated using a semiconducting crystal whose band gap is 2.0 eV. Parallel light from the pointer is shone through a photographic slide onto a $0.3\,\text{m} \times 0.3\,\text{m}$ screen 3 metres away. The slide comprises a periodic array of fine dots arranged in a square pattern which repeats with a periodicity of $a = 0.1\,\text{mm}$.



(i) Calculate the wavelength, λ, of the light generated by the laser pointer. [4 marks]

(ii) Derive an expression for the angles of the diffracted beams arising from the periodicities in the dot pattern in

(a) the direction along a slide edge

(b) the direction along the diagonal of the slide. [4 marks]

(iii) Sketch the light pattern you would expect to see on the screen, clearly indicating its numerical dimensions. How many spots will be seen in total? [4 marks]

As part of a development programme to improve the diode laser, the quality of the semiconductor crystal is studied by accelerating a beam of electrons through a voltage V, and passing it, at normal incidence, through a very thin slice of the crystal onto a 0.15 m diameter circular fluorescent screen 1 m away. The crystal has cubic symmetry, with a lattice constant of 0.56 nm, and the slice is cut so that its faces are parallel to the faces of the crystal's unit cell.

(iv) Given that the electron velocity is much less than the speed of light, derive an expression for the electron de Broglie wavelength corresponding to the accelerating voltage V. [3 marks]

(v) The voltage accelerating the electron beam is varied by the researcher. Over what range of V values will the pattern on the screen consist of exactly 5 spots? [5 marks]

[TOTAL 20 marks]

6. (i) The electron has intrinsic spin $S = 1/2$. The spin operators can be represented by Pauli matrices:

$$\hat{S}_x \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In this representation of spin operators, an eigenstate of the operator is an eigenvector of the corresponding matrix. For instance, the eigenvalues of $\hat{S}_z$ are $\pm\hbar/2$, corresponding to the eigenvectors $\binom{1}{0}$ and $\binom{0}{1}$.

More generally, a spin with amplitude α in the spin-up state (parallel to z) and amplitude β in the spin-down state (antiparallel to z) is represented by the column vector:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

(a) Show that $\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenstate of $\hat{S}_x$. Show that this corresponds to a spin polarised in the positive x-direction. [4 marks]

(b) An apparatus is designed to measure the z-component of the spin, S_z. A beam of electrons, previously polarised by a strong magnet in the $+x$-direction, passes through this apparatus. Describe what the apparatus measures for these electrons, and the spin states of the electrons after the measurement. [5 marks]

(ii) The Hamiltonian for an electron spin in a magnetic field B in the z-direction is

$$\hat{H} = 2\mu_B B \hat{S}_z/\hbar$$

where μ_B is the Bohr magneton.

(a) What are the two energy eigenstates and eigenvalues of this spin Hamiltonian? How does each of the eigenstates evolve in time? [4 marks]

(b) An electron spin is prepared so that it is polarised in the $+x$-direction. At time $t = 0$, this spin is placed in a magnetic field B in the z-direction. Write down the vector representing the spin state at subsequent times $t > 0$. [3 marks]

(c) The x-component of this spin is measured at time $t > 0$. Show that the expectation value of the result of this measurement is:

$$\langle S_x \rangle = \frac{\hbar}{2}\cos\left(\frac{2\mu_B B t}{\hbar}\right).$$

[4 marks]

[TOTAL 20 marks]

7. (i) Write brief notes on the production of radiation from a laser. Outline the characteristics of the output radiation from a laser compared to that from a spontaneously emitting source.

[8 marks]

(ii) A hydrogen maser can be used as a high precision atomic clock. (A maser is the microwave equivalent of a laser.) Each atom in a hydrogen maser emits at a frequency f_0 in the rest frame of the atom. This can be regarded as the "ticking" of a clock.

(a) For a single atom moving transverse to the line of sight at velocity v, write down the relativistic expression for the observed frequency, f_L, in the laboratory frame.

Show that the change in frequency between the two frames is given approximately by

$$\Delta f = f_L - f_0 \approx -\frac{f_0 v^2}{2c^2}$$

where it is assumed that $v^2 \ll c^2$.

(b) For the gas in the maser, at a uniform temperature T, show that the fractional change in frequency, for atoms having the mean thermal energy, can be written as

$$\frac{\Delta f}{f_0} = -\frac{3kT}{2M_H c^2}\,.$$

[Note that the *first order* Doppler effect cancels under these circumstances.]

(c) The maser in an atomic clock is operated at a temperature of around 300 K. Calculate the fractional change in frequency. In many applications, frequencies are compared to 1 part in 10^{15}. In order to make use of this level of accuracy how stable must the temperature be?

[8 marks]

(iii) Hydrogen maser atomic clocks have been used to test the equivalence principle by comparing a clock on the ground with one in a spacecraft orbiting at an altitude of 10^4 km. Outline what physical principle is being tested and why this procedure provides such a test.

[4 marks]

[TOTAL 20 marks]

8. (i) A photon is emitted from a nucleus which exists in a free state in a perfect gas. While this emission is associated with a transition between two distinct quantised energy levels, the photon can have a range of possible energies. In the context of ${}_{77}\mathrm{Ir}^{191}$, for which details are given below, explain and compare the magnitudes of the contributions to the spread in and changes to photon energies at varying pressures and temperatures, including the recoil shift.

[6 marks]

(ii) One way to detect such a photon is by reabsorbing it in the same material with the same transition which was responsible for the emission. Explain where difficulties with this 'resonance' approach will lie.

(Hint: consider conservation of momentum in the absorption and emission processes.)

[2 marks]

(iii) In principle, the material could be put into a centrifuge to overcome such difficulties. Calculate the rotational velocity a centrifuge of radius 20 cm would need to attain in the case of ${}_{77}\mathrm{Ir}^{191}$.

[4 marks]

(iv) Discuss whether the proposal in (iii) is plausible.

[2 marks]

(v) Explain how, if the nucleus is embedded in a lattice, recoil can be taken up by large numbers of atoms.

[3 marks]

(vi) A photon falling height x in a gravitational field has its frequency shifted by $\delta\nu/\nu = gx/c^2$. How could one measure this?

[3 marks]

[TOTAL 20 marks]

Assume that ${}_{77}\mathrm{Ir}^{191}$ emits a gamma at 100 keV, and has a lifetime of 10^{-10} s.
The Ir nucleus has a mass of $940 \times 191\,\mathrm{MeV/c^2}$.
Planck's constant $/2\pi$ is equivalent to $\hbar = 6.58 \times 10^{-22}\,\mathrm{MeV\,s}$.
Boltzmann's constant is equivalent to $k_B = 8.62 \times 10^{-11}\,\mathrm{MeV/K}$.
Assume the temperature is 300 K.

9. (i) Beginning with the 1st Law of Thermodynamics:

$$dU = đQ + đW \tag{9.1}$$

where U = internal energy, Q = thermal energy and W = work done on a system, use the equation of state and the fact that $c_p - c_v = R/M$ to derive the following expression for the entropy of a volume of air (considered to be a perfect gas) containing unit mass (so that density, $\rho = 1/V$ where V = volume):

$$S = c_p \ln T - (R/M) \ln p + S_0 \,. \tag{9.2}$$

In equation (9.2):
c_p = specific heat at constant pressure = 1005 J kg^{-1} K^{-1},
S = entropy,
R = universal gas constant = 8.314×10^3 J kmol^{-1} K^{-1},
M = average molecular weight of dry air = 28.96,
p = pressure, and
S_0 = constant of integration.

[10 marks]

(ii) If a bubble of air in the atmosphere is heated (e.g. by the release of latent heat) and as a result rises from the pressure level $p_1 = 7 \times 10^4$ Pa (where $T_1 = 284$ K) to $p_2 = 5 \times 10^4$ Pa ($T_2 = 270$ K), calculate the change in entropy $\Delta S = S_2 - S_1$ that occurs.

[4 marks]

(iii) Consider the Earth in space. A quantity of incoming solar energy, ΔQ_{in}, is absorbed by each m^2 of surface area of the Earth/atmosphere system every second. The Earth balances this in equilibrium by emitting thermal radiation, ΔQ_{out}, back to space. Use simple considerations to show that the influence of the Earth is to increase the entropy of the radiation field. How, physically, is the corresponding increase of disorder manifested?

[6 marks]

[TOTAL 20 marks]

10. Write an essay about ONE of the following subjects.

(i) If there are still uncertainties over the cause of global warming, should we even be contemplating measures to reduce greenhouse gas emissions, if these measures could harm our economy?

(ii) The physics of renewable energy.

(iii) Asteroids.

(iv) The value of a training in physics for those entering other careers.

(v) Following the space shuttle Columbia disaster, what justification is there for future manned space flight missions?

(vi) "I know of no one who has participated actively in the advance of physics in the postwar period whose research has been significantly helped by the work of philosophers."
(*Steven Weinberg*)

[20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

PHYSICS II Comprehensive Paper

For Third-and Fourth-Year Physics Students

Friday 30th May 2003: 10.00 to 13.00

Answer FIVE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

1. Synchronous Lateral Excitation of the deck of the central span of the London Millennium bridge can be modelled using simple ideas of 1D damped simple harmonic motion.

(i) Write down a differential equation which describes the *free* damped motion of a mass, M, connected to a spring of stiffness K, subject to a viscous damping force of the form, $\alpha \times velocity$. [1 mark]

Tests showed that the deck swayed from side to side with a natural frequency of 0.5 Hz. On the opening day the bridge was observed to sway at the natural frequency with a maximum displacement of 70 mm.

(ii) Determine the maximum velocity of the motion on the opening day assuming it to be harmonic. [2 marks]

The original bridge design set the viscous damping coefficient to be 1% of the critical value, $\alpha_{critical}$, for the known stiffness, K, and the design load of the bridge.

(iii) Show that the critical damping coefficient is given by

$$\alpha_{critical} = \sqrt{4KM} = 2\omega_o M$$

where ω_o is the natural angular frequency of the bridge. [4 marks]

The excessive lateral motion is caused by pedestrians who, in trying to keep their balance on the moving deck, make synchronous sideways steps in the direction of the motion. At each such step a person exerts a lateral force $F = Pv$ in the same direction as the lateral velocity of the deck, v. Tests show that the motion just starts to amplify when about 150 $\sim$70 kg persons walk along the deck in this manner. If the number of pedestrians is >150 then the deck motion becomes effectively negatively damped and then the amplitude increases with time .

(iv) Find an expression for the coefficient, P in terms of $\alpha_{critical}$ [2 marks]

(v) Assuming that 300 people begin to walk in step at an initial lateral motion amplitude of 1 mm, estimate the time taken for the motion to build up to an amplitude of 70 mm. [5 marks]

Several remedies for the excessive lateral motion were studied.

(vi) Suggest why the simple idea of increasing the lateral stiffness of the bridge to increase the natural frequency to 1.5 Hz would have required extensive rebuilding and the probable destruction of the original *thin* design concept. [2 marks]

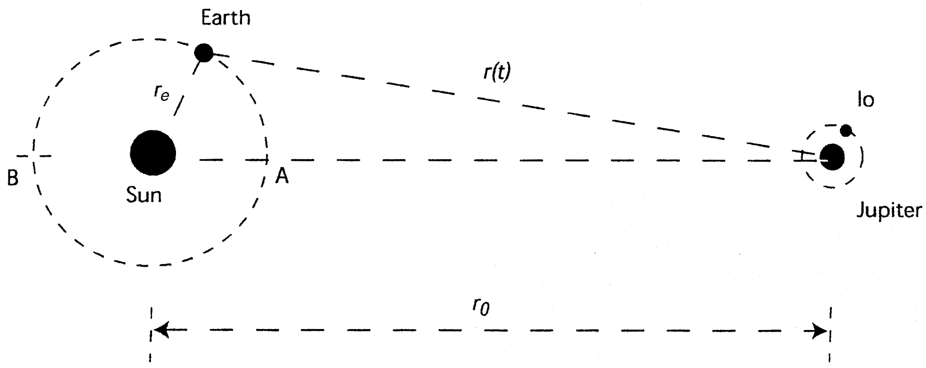
The best and, it turns out, successful remedy was to introduce very much more viscous damping. Tests showed that the synchronous excitation from pedestrians grows until a density of 2 persons per m^2 is attained. At higher densities the crowd cannot move in step.

(vii) Estimate the damping coefficient required to ensure no amplification up to a crowd density of 2 persons per m^2 . [4 marks]

The deck of the central span has a mass, $M = 2.88 \times 10^5$ kg, a length of 144 m and a width of 4 m.

[TOTAL 20 marks]

2. In 1676 Roemer measured the speed of light c by observing apparent fluctuations in the orbital period of Jupiter's satellite, Io. The figure shows the Earth in a circular orbit about the Sun with radius r_e, the Jupiter/Io system being distant r_0 from the Sun. Assume that the orbital planes of the Earth and Io coincide, and that during an Earth year ($= T$) the Jupiter/Io system effectively does not move with respect to the Sun. At $t = 0$ the Earth is at A.



(i) Derive an approximate expression for $r(t)$, the Earth/Jupiter distance at time t. [5 marks]

(ii) By noting that the period of Io's orbit, Δt, is much less than T, show that during Δt the change in r is given approximately by

$$\Delta r \approx \frac{2\pi r_e \Delta t}{T} \sin\left(\frac{2\pi t}{T}\right),$$

and write down an expression for the resulting difference between the *observed* period of Io's orbit, and its *actual* period, Δt. [4 marks]

(iii) Using the expression from part (ii), show that the sum of the observed delays after Io has made n orbits (starting from $t = 0$) is given by

$$\text{(accumulated delay)} \approx \frac{r_e}{c}\left[1 - \cos\left(\frac{2\pi n \Delta t}{T}\right)\right].$$

When the Earth reaches point B, is this expression in accordance with your expectation? Give a reason. Estimate the speed of light if the observed accumulated delay when the Earth first reaches B is 16 min. [3 marks]

(iv) Due to the use of unreliable clocks at the time, the uncertainty in the observed accumulated delay was 5 min. Also, the uncertainty in r_e was 20%. What was the induced uncertainty in the resulting estimate of c?

[3 marks]

(v) Describe qualitatively how, over the six month period during which the Earth moves from A to B, the estimate of c made in part (iii) might be improved upon.

[5 marks]

[TOTAL 20 marks]

Data:
$\Delta t = 42.5$ h
$r_e = 1.5 \times 10^{11}$ m,
$r_0 = 7.78 \times 10^{11}$ m.

3. (i) One of the ways in which X-rays of 10 to 100 keV energy interact with matter is by scattering through the Compton effect. With the use of a clearly labelled diagram, describe the Compton process in such an interaction and write down the equations for the conservation laws that apply. [5 marks]

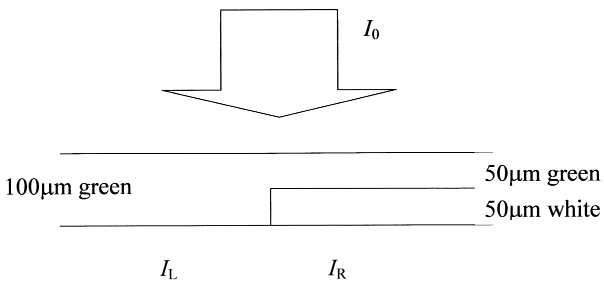
(ii) The high voltage tubes used to generate X-rays in this energy range use fast electrons from the cathode to hit a pure metal target anode. This produces X-rays with a continuous spectrum, with a well defined maximum energy, and narrow emission lines at energies characteristic of the metal target. Outline the processes which produce these features of the output X-ray spectrum. [6 marks]

(iii) A narrow beam of monoenergetic X-rays has an initial intensity I_0. The beam intensity reduces by dI in passing through a length dx of material. In the material there are N atoms per unit volume, each with absorption cross section σ. Show that at some distance, x, into the material the beam intensity I along the beam direction is given by

$$I = I_0 e^{-\mu x}$$

where μ is the linear attenuation coefficient, and find an expression for μ. [5 marks]

(iv) Works of art are often X-rayed to investigate what other paint layers lie beneath the visible surface. The X-ray film is placed against the back of the canvas to record the X-ray image. In old paintings, many of the pigments contain metals and these dominate the X-ray absorption by the paint. The diagram below shows a cross section through a small area of a painting, illuminated by a monoenergetic X-ray beam of intensity I_0. The emergent intensity I_L is $\frac{1}{2}I_0$. For the two pigments $\mu_{white}/\mu_{green} = 3$. What is I_R as a fraction of I_0?



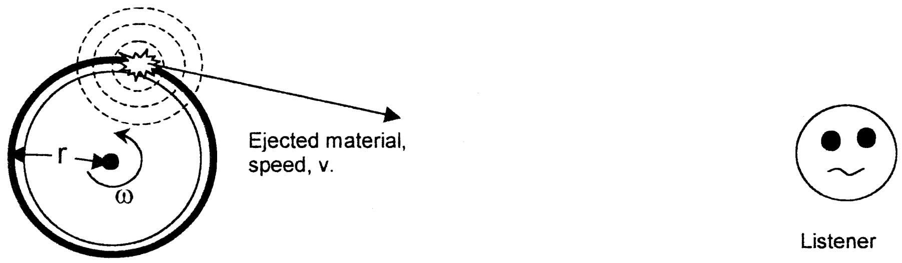
[4 marks]

[TOTAL 20 marks]

4. (i) Define the term "moment of inertia", I, as applied to a rigid body rotating about a fixed axis. Write down a general expression from which it can be calculated, and also an expression relating applied torque and the angular acceleration of a rotating body.

[2 marks]

A "Catherine wheel" firework consists of a mass M of combustible material evenly distributed around a wheel-shaped framework, whose mass, μ, is concentrated at the same radius, r. When at rest it is lit, and gradually ejects the combustible mass tangentially from its rim at a high velocity, v. The burning region is designed to emit a piercing whistle at a frequency $f_0 = 1000\,\mathrm{Hz}$.



(ii) By considering the tangential ejection of an infinitesimal mass at the wheel's perimeter, and assuming that at all times the angular velocity of the wheel, ω, is much less than v/r, justify the equation

$$I\frac{d\omega}{dt} = rv\frac{dm}{dt}$$

where m denotes the mass of material already burnt.

[3 marks]

(iii) Derive an expression for the corresponding angular acceleration of the wheel at an instant when a mass m of the combustible material has already been burnt.

[3 marks]

(iv) Assuming that the wheel is stationary at the outset, and neglecting friction, integrate your answer for (iii) to derive an expression for the final angular velocity, ω_f, of the wheel.

[4 marks]

(v) Evaluate ω_f, for a wheel of radius 0.1 m, comprising 0.02 kg of combustible material, ejected at $50\,\mathrm{m\,s^{-1}}$, from a framework weighing 0.2 kg.

[2 marks]

(vi) Denoting the speed of sound in air as c, write down an expression for the frequency f of the sound wave you would hear from a source moving towards you at a speed u, and emitting sound at a frequency f_0.

[2 marks]

(vii) A listener stands many metres away in the plane of rotation of the wheel. In as much detail as possible, assuming that the firework burns for about 10 seconds, describe what they would hear.

[4 marks]

[TOTAL 20 marks]

Data:
speed of sound, $c = 330\,\mathrm{m\,s^{-1}}$

5. *Small angle approximations should be used throughout this question.*

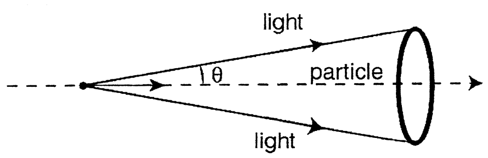
(i) A spherical mirror has radius of curvature R.

(a) By considering parallel rays incident on the mirror at small distances from the optic axis, show that the focal length, f, of the mirror is $f = R/2$.

(b) Two light beams which intersect at an angle 2θ are directed towards the same mirror. Each beam is made up of parallel rays. Show, using a diagram, that the foci of the two beams are separated by a distance $R\theta$. [7 marks]

(ii) Write down the relationship between energy, E, and momentum, p, for a relativistic particle of mass m, and the relationship between p and the velocity, v. [2 marks]

(iii) Cerenkov radiation is visible light emitted when a charged particle travels at a velocity greater than the speed of light in a transparent material. It is emitted from each point on the particle trajectory on the surface of a cone around the particle direction as shown in the diagram.

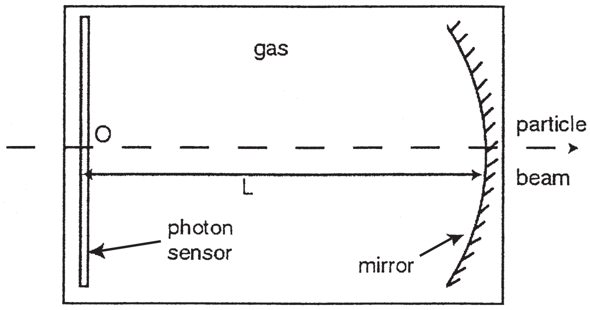


If the material has refractive index n, then $\cos\theta = 1/\beta n$, where $\beta = v/c$.

(a) What is the speed of light in the material?

(b) The refractive index of a gas can be written $n = 1 + \delta$, where $\delta \ll 1$. If $1 - \beta \ll 1$, show that light is emitted only when $p > \dfrac{mc}{\sqrt{2\delta}}$. [5 marks]

(iv) A Cerenkov detector for identifying particles of $\beta \approx 1$ in a beam at CERN is constructed from a large sealed container of gas containing a spherical mirror and a planar photon sensor as shown in the diagram. The refractive index of the gas is $n = 1.0002$. The beam passes along the optical axis of the mirror. The path length L is 2.5 m and the radius of the mirror is 5 m.



(a) Calculate the threshold momentum in GeV/c for emission of Cerenkov radiation by a π meson and by a proton.

(b) Now consider Cerenkov radiation from a $\beta = 1$ particle. A light ray originates from point O. At what radial distance from the beam does it strike the photon sensor?

(c) What pattern does the light intensity distribution on the surface of the photon sensor have?

[6 marks]

[TOTAL 20 marks]

Mass of π meson = 140 MeV/c^2
Mass of proton = 938 MeV/c^2

6. A plane monochromatic electromagnetic wave has electric field $\mathbf{E} = \hat{\mathbf{e}} E_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$. The direction of the field is given by the unit vector $\hat{\mathbf{e}}$, the three-dimensional wavevector is $\mathbf{k}$, the wave frequency is ω, and position and time are $(\mathbf{r}, t)$. The wave is incident on a free particle of mass m and charge q.

(i) Write down the *non-relativistic* equation of motion for the particle. [2 marks]

(ii) What is the time-averaged root mean square acceleration of the particle, assuming it moves a negligible part of a wavelength during one oscillation? [5 marks]

(iii) What is the order of magnitude of the maximum field amplitude $E_0 \sim E_{\rm max}$ for which this non-relativistic treatment is justified? [3 marks]

(iv) What is the corresponding magnitude and direction of the magnetic field component of the electromagnetic wave? [2 marks]

(v) If we consider the full relativistic case, $E_0 \gg E_{\rm max}$, what physical effects will no longer be negligible? Write down the full relativistic equation of motion. [3 marks]

(vi) Qualitatively, how will the motion differ from the non-relativistic case? [3 marks]

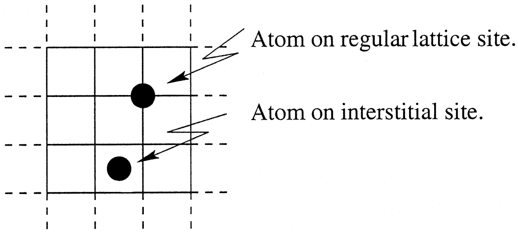
(vii) Find the value of E_0 for an electron in the case of visible light of wavelength 500 nm. [2 marks]

[TOTAL 20 marks]

7. (i) Define the entropy S of a system in terms of the number of possible microstates Ω associated with a given macrostate.

[2 marks]

In a monatomic solid, each atom can occupy either a regular lattice site or an interstitial site. Assume the energy of an atom situated on a lattice site is zero and that the energy of an atom on an interstitial site is $\epsilon > 0$. The number of atoms is N. The number of regular sites and the number of interstitial sites are also both equal to N.



(ii) Show that the entropy $S(n)$ of the crystal when n of the N atoms are at interstitial sites is

$$S(n) = 2k_B \left[\ln N! - \ln n! - \ln(N-n)!\right],$$

where k_B is Boltzmann's constant.

[3 marks]

Let E denote the energy of the system and define the temperature T by

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{\partial S}{\partial n}\frac{\partial n}{\partial E}.$$

(iii) Use Stirling's formula

$$\ln x! \approx x \ln x - x \qquad \text{for } x \gg 1$$

to show that

$$\frac{1}{T} = \frac{2k_B}{\epsilon}\left[\ln\left(\frac{N-n}{n}\right)\right].$$

[4 marks]

(iv) (a) Use the above equation to find the fraction of atoms occupying interstitial sites, $\frac{n}{N}$, as a function of temperature T.

[3 marks]

(b) What is the ratio $\frac{n}{N}$ in the limit $T \to 0$? Describe the associated microscopic state(s) and find the entropy.

[4 marks]

(c) What is the ratio $\frac{n}{N}$ in the limit $T \to \infty$? Describe the associated microscopic state(s) and find the entropy.

[4 marks]

[TOTAL 20 marks]

8. (i) A particle of mass m in a parabolic well in one dimension is governed by the Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \qquad \text{where} \qquad V(x) = \frac{1}{2}m\omega^2 x^2.$$

The ground state wavefunction has the form:

$$u_0(x) = N\exp(-x^2/2\sigma^2) \qquad \sigma = (\hbar/m\omega)^{1/2}.$$

(a) Explain briefly the motion of this particle in classical mechanics, indicating the physical significance of the quantity ω. [4 marks]

(b) Sketch on the same diagram the potential $V(x)$ and the form of the ground state wavefunction, $u_0(x)$. [2 marks]

(c) State the Heisenberg uncertainty principle. Hence, or otherwise, find the typical magnitude (*e.g.*, root-mean-square value) of the momentum of a particle in the ground state, in terms of m and ω. (Do not worry about numerical factors.) [4 marks]

(d) A particle is in the ground state at time $t < 0$. The potential $V(x)$ is suddenly reduced to zero at time $t = 0$. A calculation shows that the wavepacket spreads out. Its width X at time t obeys the equation:

$$X^2(t) = X^2(t=0) + \frac{\hbar\omega t^2}{m}.$$

Sketch $X(t)$ as a function of time. Use simple arguments to explain the behaviour at large t. [4 marks]

(ii) A gas of ultracold bosonic atoms (mass m) is trapped in a three-dimensional potential well $V(\mathbf{r})$:

$$V(\mathbf{r}) = \frac{m}{2}\left(\omega^2 x^2 + \Omega^2 y^2 + \Omega^2 z^2\right), \qquad \text{with} \quad \Omega \gg \omega.$$

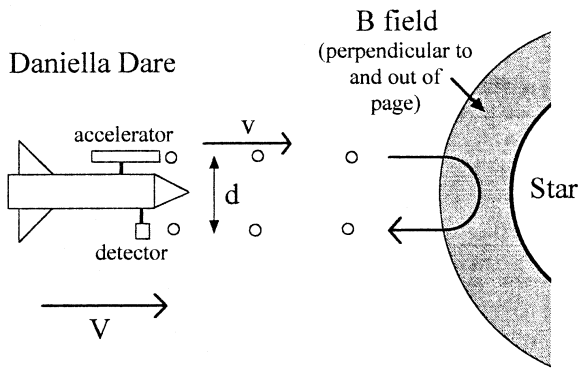
You can assume that the atoms do not interact with each other, and that the temperature is at absolute zero. (You may use results from part (i).)

(a) This cloud of atoms forms a cigar-shaped ellipsoid. Explain why, giving the orientation of this ellipsoid. [4 marks]

(b) The trapping potential is switched off suddenly. The cloud is imaged after a delay. Describe how the observed shape of the cloud depends on the time of the measurement. [2 marks]

[TOTAL 20 marks]

9. Captain Daniella Dare, astrophysicist of the future, is out exploring the galaxy in her relativistic space-ship. She approaches an unknown type of star and wants to find out the strength of the magnetic field at the surface.



(i) (a) Daniella has a particle accelerator which accelerates nuclei of charge q and rest mass m by an electric field E applied for time t. How are p and E related? (ignore any magnetic effects). [2 marks]

(b) What is the relativistic momentum p of the nuclei in the ship's reference frame? [2 marks]

(c) Using the answer to parts (a) and (b) show that:

$$v = c\sqrt{\frac{(qEt/mc)^2}{1+(qEt/mc)^2}}.$$

[5 marks]

(ii) Daniella's space-ship is approaching the star at relativistic velocity V. As a result the nuclei have velocity v' in the star's reference frame. Use:

$$dx' = \gamma(dx - Vdt) \qquad dt' = \gamma(dt - Vdx/c^2)$$

to show that:

$$v' = \frac{v - V}{1 - Vv/c^2}.$$

[5 marks]

(iii) Assume that above a certain distance from the star's surface the magnetic field is zero. Below this distance it has a constant value of B. For the beam of nuclei to reach the detector on Daniella's space-ship the nuclei must enter the magnetic field and make a 180 degree turn though a radius $d/2$ where d is the distance between the accelerator and detector on Daniella's ship (assume that the accelerator and detector are aligned relative to the B field so that this may occur). By finding the cyclotron resonant frequency ω_c, show that the nuclei are detected if:

$$B = \frac{2m\gamma(v')v'}{qd}.$$

If $v' = 0.5c$ and the nuclei have $Z = 118$, $A = 260$ and $d = 20\,\mathrm{m}$, calculate B. [6 marks]

[TOTAL 20 marks]

10. Write short notes on FIVE of the following topics

(i) The photoelectric effect.

(ii) Black body radiation.

(iii) The two-slit experiment and quantum mechanics.

(iv) The Bohr atom.

(v) The origin of tides.

(vi) Entropy.

(vii) Virtual particles.

(viii) The equivalence principle.

(ix) The source of the Sun's energy.

(x) Quantum computing.

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ASTROPHYSICS

For Third-and Fourth-Year Physics Students

Wednesday 21st May 2003: 10.00 to 12.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) A star of mass M_s and planet of mass M_p orbit the centre of gravity of the system. Denoting the position vector of the planet by $\mathbf{r_p}$, and for a coordinate system with origin the centre of mass, show that the acceleration of the planet is given by $\ddot{\mathbf{r}}_\mathbf{p} = -(GM'_s/r_p^2)\ \hat{\mathbf{r}}_\mathbf{p}$, where $M'_s = M_s^3/(M_s + M_p)^2$.

[4 marks]

(ii) For circular orbits, derive expressions for the orbital speed of the planet v_p, and of the star v_s, and the orbital period T, in terms of $r_t = r_p + r_s$, where r_p and r_s are the radii of orbit of the planet and star respectively.

[5 marks]

(iii) By summing the kinetic and potential energy of the system confirm that the virial theorem is satisfied.

[5 marks]

(iv) The habitable zone (where water is liquid) is at a distance close to 1AU for a planet orbiting a star like the Sun. Radial velocity searches can detect the motion of stars with orbital speeds as small as $10\mathrm{m\,s^{-1}}$. What is the smallest mass planet in the habitable zone about a Sun-like star that could be detected by such a search? Express your answer in units of the mass of the Earth, $M_\oplus$.

[6 marks]

[TOTAL 20 marks]

1AU$= 1.50 \times 10^{11}\,\mathrm{m}$
$G = 6.67 \times 10^{-11}\mathrm{N\,m^2\,kg^{-2}}$
$M_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$
$M_\oplus = 5.98 \times 10^{24}\,\mathrm{kg}$

2. (i) State the three fundamental assumptions made in applying the virial theorem to the measurement of mass in galaxy clusters.

[3 marks]

(ii) A galaxy cluster is modelled as spherically symmetric with a power-law density profile $\rho = Ar^{-\alpha}$, out to a cutoff radius R. Compute the total mass M in terms of A and R.

[3 marks]

(iii) By integrating in shells show that the expression for the potential energy of the galaxy cluster is $U = -\dfrac{(3-\alpha)GM^2}{(5-2\alpha)R}$.

[7 marks]

(iv) Measurements of the light distribution in a galaxy cluster indicate a value $\alpha = 1.5$. By using the virial theorem show that the total mass of the cluster can be expressed as $M = 4\sigma^2 R/G$, where σ is the one-dimensional velocity dispersion of the galaxies in the cluster.

[4 marks]

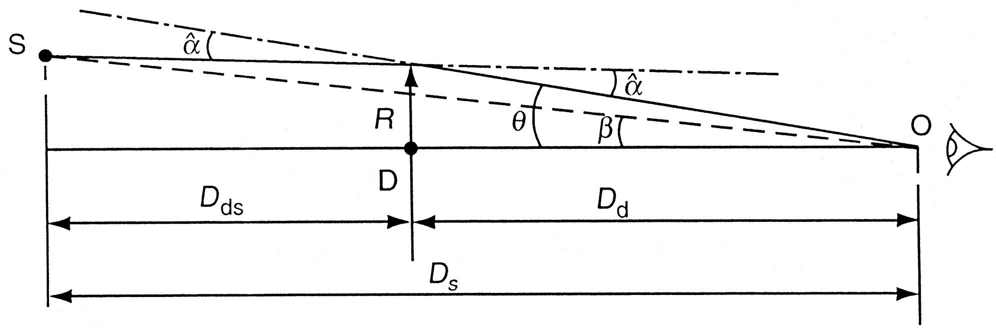
(v) The cluster has a radius $R = 4\,\text{Mpc}$, and the measured one-dimensional velocity dispersion is $\sigma = 1150\,\text{km}\,\text{s}^{-1}$. Compute the mass of the cluster in units of solar mass.

[3 marks]

[TOTAL 20 marks]

1pc$= 3.09 \times 10^{16}\,\text{m}$
$M_\odot = 1.99 \times 10^{30}\,\text{kg}$
$G = 6.67 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2}$

3. (i) Light from a distant astronomical source S is deflected by a galaxy D, and is seen by the observer O at a displaced angular position, as shown in the diagram below. Assuming all angles are small, the lens equation relating the angles β (the true source position), θ (the observed position), and $\hat{\alpha}$ (the deflection angle) is $\beta = \theta - \hat{\alpha} D_{ds}/D_s$. Treating the galaxy as a point mass, and using the fact that the deflection angle at impact parameter R is $\hat{\alpha} = 4GM/(c^2 R)$, rewrite the lens equation as a quadratic expression in the variable θ. Using the substitution $\theta_E^2 = \dfrac{D_{ds}}{D_s D_d}\dfrac{4GM}{c^2}$, show that there are two solutions for θ, with primary $(+)$ and secondary $(-)$ images at angles $\theta_\pm = \left[\beta \pm \left(\beta^2 + 4\theta_E^2\right)^{1/2}\right]/2$ [4 marks]



(ii) What is the typical mass of a large spiral galaxy? For a quasar lensed by a galaxy, the distances D_{ds}, D_s, D_d, are of order 1Gpc. Calculate the typical size of the Einstein angle θ_E for this configuration, in arcsec. [4 marks]

(iii) A double quasar source S consists of a bright component S1 at angle β, and a fainter component S2 located at larger angle $\beta + d\beta$, along L, the line on the sky joining the galaxy D and S1. Four images are formed along L, the two primary images $I1_+$, $I2_+$, and the two secondary images $I1_-$, $I2_-$. Using the above solution for $\theta_\pm$, explain carefully the relative positions of the four images along L, and draw a diagram of the configuration on the sky showing D, S1, S2, $I1_+$, $I2_+$, $I1_-$, $I2_-$. [7 marks]

(iv) Explain the formation of images of +ve and −ve parity. [5 marks]

[TOTAL 20 marks]

1 pc$= 3.09 \times 10^{16}$ m
$M_\odot = 1.99 \times 10^{30}$ kg
$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}
$c = 3.00 \times 10^8$ m s^{-1}

4. (i) Starting with the relation between gas pressure P and temperature T for a mixture of species of number density n, $P = \Sigma_i n_i k_B T$, derive the equation of state for a fully ionised pure hydrogen gas, and the mean molecular weight.

[3 marks]

(ii) Using the virial theorem, briefly describe the collapse of a gas cloud to form a star, up to the point of nuclear ignition, by reference to the time dependence of the total energy, the kinetic energy, and the potential energy.

[4 marks]

(iii) Treating the Sun as a uniformly dense sphere of ionised hydrogen of mass $M_\odot$, and radius $R_\odot$, the potential energy is given by $U = -3GM_\odot^2/(5R_\odot)$. Use the virial theorem to derive an expression for the average kinetic energy density. Using the fact that the pressure in a non-relativistic gas is $2/3$ the kinetic energy density, estimate the average temperature in the Sun.

[5 marks]

(iv) For matter in thermal equilibrium at temperature T, the energy density of radiation is $u = 4\sigma T^4/c$, where σ is the Stefan-Boltzmann constant. Estimate the total energy in radiation in the Sun. Using the measured luminosity of the Sun $L = 3.83 \times 10^{26}\,\mathrm{W}$, estimate the average time it takes a photon to escape from the interior of the Sun.

[5 marks]

(v) Photons of mean free path λ require some $N = R_\odot^2/\lambda^2$ steps to random walk out of the Sun. Estimate the mean free path of photons in the Sun.

[3 marks]

[TOTAL 20 marks]

$M_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$
$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
$m_p = 1.67 \times 10^{-27}\,\mathrm{kg}$
$k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
$R_\odot = 6.96 \times 10^{8}\,\mathrm{m}$
$\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$

5. (i) Derive the equation of hydrostatic equilibrium, and show that the pressure at the centre of a constant density sphere of mass M and radius R, is given by $P_c = 3GM^2/(8\pi R^4)$. [6 marks]

(ii) A white dwarf may be treated as a uniformly dense sphere, permeated by a gas of free electrons of number density n_e. In a low mass white dwarf the distribution of electron energies in a volume V is given by the non-relativistic distribution function $n(E)dE = CVE^{1/2}dE$, up to the Fermi energy E_F, where C is a constant. Here $n(E)dE$ is the number of electrons in the volume with energies between E and $E + dE$. Obtain an expression for the Fermi energy in terms of n_e and C. [3 marks]

(iii) Show that the kinetic energy density of degenerate electrons is proportional to $n_e^{5/3}$. The electron degeneracy pressure supports the star. Using the fact that the pressure in a non-relativistic gas is $2/3$ the kinetic energy density, show that the radius of a white dwarf is proportional to $M^{-1/3}$. [5 marks]

(iv) A white dwarf is accreting mass from a companion. As the mass increases the electrons become relativistic. The expression for the degeneracy pressure in the ultra relativistic limit is $P \propto n_e^{4/3}$. Explain by means of a plot of R against M, and by careful argument, the behaviour of the white dwarf as the ultra relativistic limit is approached. [6 marks]

[TOTAL 20 marks]

6. (i) A type 1a supernova explodes in a distant galaxy, and is observed to reach peak brightness $m = 15.5$. If the absolute magnitude at peak brightness is $M = -19.4$, compute the distance d in Mpc.

[4 marks]

(ii) Before the supernova explosion the brightness of the galaxy was $m = 14.2$. Compute the ratio of the flux from the galaxy over the flux from the supernova at maximum brightness.

[3 marks]

(iii) In the spectrum of the galaxy the Ca absorption line of restframe wavelength $\lambda_\circ = 393.37$nm, is measured at a wavelength $\lambda = 400.92$ nm. What is the redshift z of the galaxy? Using the Hubble law $cz = H_\circ d$, estimate the Hubble constant $H_\circ$ in units of $\mathrm{km\,s^{-1}\,Mpc^{-1}}$. [3 marks]

(iv) A physical model of the supernova explosion indicates that at peak brightness the expanding photosphere of the explosion would have a diameter of 10^{13}m. What angular size does this correspond to? Express your answer both in radians and arcsec. An astronomer proposes to build a radio interferometer to resolve the supernova at a wavelength of 0.2m. What baseline would be required? Comment on the feasibility of the experiment.

[5 marks]

(v) *Briefly* outline the most important steps involved in measuring distances in the Universe, on scales larger than the solar system.

[5 marks]

[TOTAL 20 marks]

1pc$= 3.09 \times 10^{16}$ m
$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
$R_\oplus = 6.38 \times 10^{6}$ m

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

INSTRUMENTATION

For Third-and Fourth-Year Physics Students

Thursday 22nd May 2003: 10.00 to 12.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

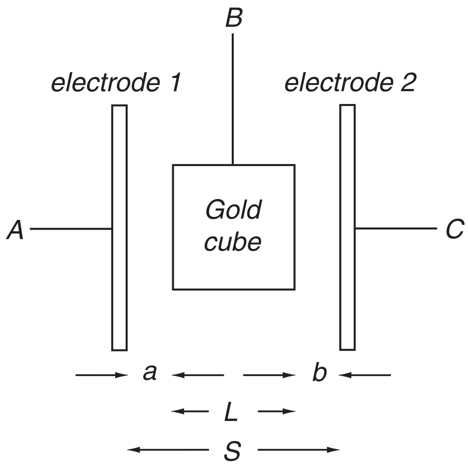
USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. A gold cube is placed between two flat copper electrodes as shown in the figure below. The position of the cube is to be measured by the capacitance between the cube and the two plates. Electrical connections are made to the two electrodes using the wires A and C. Wire B provides a very fine contact to the cube.



(i) Using a simple parallel plate capacitor formula, write down expressions for the capacitance between the cube and electrode 1, and between the cube and electrode 2. Show that if point A is left disconnected, point C is grounded, and a reference capacitor, C_{ref} is connected between point B and a sinusoidal voltage source of amplitude, V_{ref}, that the voltage signal appearing at point B is

$$V_B = V_{ref} \frac{bC_{ref}}{\left(bC_{ref} + \epsilon_o L^2\right)} .$$

[5 marks]

(ii) An alternative way of connecting the sensor is for point C to be connected to ground, nothing connected to point B, and the sinusoidal voltage, V_{ref}, applied at point A. Show that, in this case the voltage appearing at point B is

$$V_B = V_A \times \frac{b}{S - L} .$$

[5 marks]

(iii) Sketch, on the same plot, the formulae for V_B as a function of b in parts (i) and (ii) for $L = 5\,\text{cm}$, $S = 10\,\text{cm}$, $C_{ref} = 10\,\text{pf}$. Discuss the relative merits of these two read-out options as revealed by these two curves.

[5 marks]

(iv) The read-out option in part (i) is chosen for a particular application. If V_{ref} is 1 volt, and a sensitivity, Δb, of 1% of the full-scale reading is required for $b \leq 1\,\text{cm}$, what voltage sensitivity must the read-out amplifier have?

[5 marks]

[TOTAL 20 marks]

$\epsilon_o = 8.85 \times 10^{-12}\,\text{F/m}$

2. A silicon detector, (^{32}Si), is set up to detect x-rays scattered from a crystal. 100 keV x-rays are incident upon a small crystal in a collimated beam. The silicon detector is 200 μm thick, square with a surface area of 1 cm^2, and can be placed anywhere on a circle of radius 20 cm from the crystal.

(i) The silicon detector is positioned such that it only registers x-rays scattered by 90°. The x-rays detected are seen to have a mean energy of 83.4 keV. It is postulated that the x-rays are scattered by the Compton process. By considering energy and momentum conservation for 90° Compton scattering, verify that this measured energy is as expected. [You should NOT use the formula given in part (iii) for this.]

[5 marks]

(ii) The photoelectric absorption cross-section in silicon for 83.4 keV x-rays is σ_{pe} given below. What is the probability that the scattered x-rays interact by the photoelectric process in the silicon detector?

[5 marks]

(iii) After Compton scattering through an angle θ the x-rays have an energy given by

$$E_f = \frac{E_i}{1 + \frac{E_i}{m_e c^2}\,(1 - \cos\theta)} \,.$$

Use this formula to estimate the energy width of the photoelectric peak seen in the silicon detector due to the geometry.

[5 marks]

(iv) Assuming the measured energy width is consistent with your calculation in part (iii) (taking into account also the detector performance) what further measurements would you do to further verify the assumption that the peak at 83.4 keV is from Compton scattering.

[5 marks]

[TOTAL 20 marks]

$m_e c^2 = 511\,\mathrm{keV}$
$\sigma_{pe} = 2.0 \times 10^{-28}\,\mathrm{m}^2$
$\rho_{Si} = 2.33 \times 10^3\,\mathrm{kg/m}^3$
$1\ \mathrm{amu} = 1.66 \times 10^{-27}\,\mathrm{kg}$

3. A cylindrical wire chamber is to be used as a detector of charged particles. The construction of the chamber is very simple with a central fine wire along the centre of a metal thin wall tube.

(i) Show that the electric field in the gap between the wire and the wall within the chamber varies as

$$E(r) = A/r$$

where A is a constant. If the outer cylinder wall is connected to ground and a voltage, V_a is applied to the wire show that

$$A = \frac{V}{\ln\left(\frac{b}{a}\right)}$$

where a is the wire radius and b is the wall radius. [7 marks]

(ii) If the chamber is filled with a gas with constant electron mobility, μ_e, show that the maximum time taken for liberated electrons to reach the anode wire is

$$t_{max} = \frac{\ln\left(\frac{b}{a}\right)}{2\mu_e V}\left(b^2 - a^2\right).$$

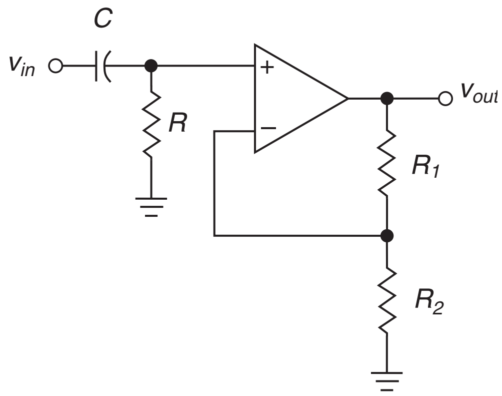
Evaluate this for a chamber with an outer diameter of 1 cm, with a 10 µm diameter wire, and with 2000 V applied between the outer wall and the wire. What does this imply about the maximum useful event rate allowed in this chamber? [5 marks]

(iii) As the operating voltage on the chamber is increased from zero, the output signal amplitude per event first starts to show amplification in the gas at 50 volts. What is the critical electric field for amplification to start in this gas? Sketch how the signal amplitude changes as the voltage is increased from zero to this value and beyond. Annotate the sketch and briefly explain the processes going on and what the relative merits are of operating detectors in different regimes. [8 marks]

[TOTAL 20 marks]

$\mu_e = 10^{-3}\,\mathrm{ms^{-1}}\left(\frac{\mathrm{V}}{\mathrm{m}}\right)^{-1}$

4.



(i) Show that the transfer function of the above circuit is

$$A(\omega) = \left(1 + \frac{R_2}{R_1}\right) \frac{j\omega CR}{(1 + j\omega CR)}$$

assuming the amplifier performs as an ideal operational amplifier. State what sort of function this circuit is performing and, *without explicitly working out the details*, sketch a Bode plot for this transfer function showing the main features. [7 marks]

(ii) Rewrite the transfer function from part (i) in an s-plane representation. Explain the significance in general of poles and zeros in the s-plane, and calculate where these are for this particular transfer function (refer back to your Bode plot where appropriate). Use the s-plane transfer function to find the response of this circuit to a step input of amplitude 1 volt at time, $t = 0$. [7 marks]

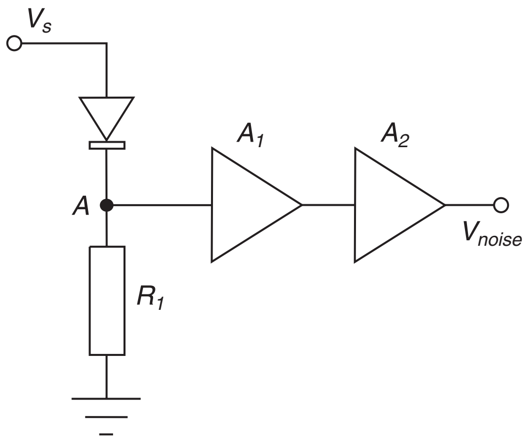
(iii) In parts (i) and (ii) we have assumed the operational amplifier is perfect. What does this mean and why do you think this amplifier has been included in this circuit? If the amplifier is not perfect, describe three effects this will have on the performance of this circuit? You do not need to include derivation of formulae in this answer, but describe in words what deviations are expected. [6 marks]

[TOTAL 20 marks]

$F(s) = \dfrac{1}{s}$ for a step input of amplitude 1.

$F(s) = \dfrac{1}{s+a}$ for $f(t) = \mathrm{e}^{-at}$

5. The circuit shown in the diagram below is designed to output voltage noise. A noise generator like this is useful for testing how other circuits respond to noise at their inputs. A_1 and A_2 are two ideal operational amplifiers used in series. The diode is operating in forward bias.



(i) Write down an expression for the mean current, I, flowing through the resistor, R_1. Show that the expected statistical fluctuation in the current, ΔI, for successive measurements each taking a time, ΔT, varies as

$$\Delta I = \sqrt{\frac{Ie}{\Delta T}}\,.$$

Discuss how this relates to the standard 'Shot noise' formula for current noise,

$$I_n = (2eIB)^{1/2}$$

where B is the measurement bandwidth. [5 marks]

(ii) Given that the Johnson noise spectral density is $\sqrt{4kTR}$ V/Hz$^{1/2}$, show that the voltage noise, V_n, at point A, due to Johnson noise and shot noise is

$$V_n = \sqrt{\left(4kTR_1B + 2eIR_1^2B\right)}$$

where T is temperature and k is Boltzmann factor. If the power supply voltage has noise on it, V_{Sn}, show how this adds into the overall noise at point A. [5 marks]

(iii) Ignoring any power supply voltage noise, V_{Sn}, show that the Johnson and Shot noise contributions are equal when

$$V_S = \frac{2kT}{e} + V_{diode}\,.$$

Both amplifiers, A_1 and A_2 have gains of 100. A_1 acts as a bandpass filter with a lower frequency break-point at 20 Hz and an upper frequency break-point at 10^4 Hz. A_2 is dc coupled with only resistive feedback elements. Resistor, R_1 is set to $10^4\,\Omega$. Work out the room temperature output noise level, V_{noise}, when V_S is (a) zero volts, and (b) 10 volts. When operating at $V_S = 10$ V what is the maximum supply voltage noise which can be tolerated to avoid increasing the overall noise by 10%? [5 marks]

(iv) State how the circuit could be improved to avoid its sensitivity to the power supply voltage noise. Two other factors which have not yet been considered are 'flicker noise' and the 'noise figure' of the amplifiers. Explain what these terms mean and discuss ways of minimising their effect. [5 marks]

[TOTAL 20 marks]

$k = 1.38 \times 10^{-23}\,\mathrm{JK}^{-1}$
$e = 1.60 \times 10^{-19}\,\mathrm{C}$

6. A photomultiplier (PMT), is used to monitor the output from a light emitting diode, which is emitting pulses at a rate of 10^4 Hz with a pulse duration of 50 ns. The mean number of photoelectrons detected in each flash of light is 1000, and these are distributed randomly throughout the 50 ns duration. The output from the PMT is connected to a 50 Ω resistor to ground and a 50 Ω coaxial cable then transports the signal 30 m to an oscilloscope.

(i) With the oscilloscope input impedance set to 50 Ω the mean signal amplitude is seen to be 1 volt and is flat over the pulse duration. Use this to verify that the gain of the PMT is close to 10^7. Given the PMT has 10 multiplication stages, work out the mean gain per stage.

[7 marks]

The individual pulses are integrated and a histogram produced of the distribution of results. This shows a peak with a full-width half-maximum of $\sim$ 8.1% of the mean peak value.

(ii) Estimate the expected contribution to the width of the distribution from (a) statistics in the photoelectron generation, and (b) variation in the secondary emission at the first dynode. Is the measured value consistent with your expectations?

[7 marks]

During modifications to the experiment it was necessary to replace the 30 m coaxial cable with three 10 m segments. Inadvertently the middle segment was fitted with a 75 Ω cable.

(iii) If the histogram referred to previously is produced again in a new experiment, what will happen to (a) the position of the peak, and (b) the full-width half-maximum? Justify your answers with calculations where appropriate. What other effects might you expect to see in (c) the histogram?

[6 marks]

[TOTAL 20 marks]

$e = 1.602 \times 10^{-19}$ C
$t_{delay} = 5$ ns/m along the coaxial cable.

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

NUCLEAR & PARTICLE PHYSICS

For Third-and Fourth-Year Physics Students

Tuesday 20th May 2003: 10.00 to 12.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. Beams of K^0 mesons can be produced in the reaction

$$\pi^- + p \to K^0 + \Lambda^0 .$$

(i) Draw a Feynman diagram showing the reaction at quark level. Label each line on the diagram clearly. Indicate which force is involved in this reaction.

[5 marks]

The wavefunctions of the K_L and K_S mesons can be written as

$$K_L^0 = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\overline{K}^0\rangle\right)$$

$$K_S^0 = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\overline{K}^0\rangle\right).$$

(ii) Show that the K_L and K_S mesons are eigenstates of CP, and determine their eigenvalues. Which force is involved in these mixings?

[5 marks]

The amplitude of a K_L beam as a function of time is given as

$$a_L(t) = e^{-im_L t} e^{-\Gamma_L t/2}$$

where m_L is the K_L mass and Γ_L is its width. Similarly, for a K_S beam

$$a_S(t) = e^{-im_S t} e^{-\Gamma_S t/2}$$

where m_S is the K_S mass and Γ_S is its width.

(iii) Write an expression for the amplitude of a K^0 beam as a function of time, $a_0(t)$, in terms of m_L, m_S, Γ_L, and Γ_S.

[5 marks]

(iv) Using your expression in part (iii) show that the intensity of K^0 remaining in an initially pure K^0 beam as a function of time is given by

$$I\left(K^0\right) = |a_0(t)|^2 = \frac{1}{4}\left[e^{-\Gamma_L t} + e^{-\Gamma_S t} + 2e^{(\Gamma_S+\Gamma_L)t/2}\cos(\Delta m\, t)\right]$$

where $\Delta m = m_S - m_L$.

[5 marks]

[TOTAL 20 marks]

$\pi^- = d\overline{u}$
$p = uud$
$K^0 = d\overline{s}$
$\overline{K}^0 = \overline{d}s$
$\Lambda^0 = uds$

2. (i) Starting with an equation for the radius of a nucleus, show that the mass density of nuclear matter is *almost* constant for all nuclei. Show that the charge density is also *nearly* constant. Use your results to estimate the ratio of electrostatic energy density for a nucleus with $A = 100$ to the rest mass energy density. *Using a typical value for the binding energy* per nucleon estimate the fractional contribution to the binding energy for this nucleus from the electrostatic energy.

[7 marks]

(ii) Write down formulae for the two contributions to the binding energy which depend on the nuclear charge, and explain, *without derivation*, their physical origins. Derive a formula for the optimum value of Z, for a fixed value of A, that results from the combined action of these two terms.

[7 marks]

(iii) The table below shows the activity properties of a number of nuclei with $A = 101$. Explain the general behaviour shown in the table in the context of your answer to part (ii).
[Note:- your prediction from part (ii) may be 1 unit out as it ignores some more subtle effects.]

[6 marks]

[TOTAL 20 marks]

Z	N	Activity	Lifetime
42	59	β^-	15 minutes
43	58	β^-	14 minutes
44	57	None	Stable
45	56	ec	3.3 years
46	55	β^+	8.5 hours
47	54	β^+	11 minutes

$m_n = 1.67495 \times 10^{-27}\,\text{kg} \equiv 939.73\,\text{Mev}/c^2$
$e = 1.602 \times 10^{-19}\,\text{C}$
$a_c = 0.7\,\text{MeV}$
$a_a = 23.3\,\text{MeV}$
$r_o = 1.3 \times 10^{-15}\,\text{m}$
$\epsilon_o = 8.85 \times 10^{-12}\,\text{As/Vm}$

3. Pairs of B mesons can be produced in e^+e^- collisions. For example

$$e^+e^- \to B^0\overline{B}^0.$$

(i) Draw a Feynman diagram of the above decay at the quark level being careful to label each line of the diagram. What forces are involved in this reaction?

[4 marks]

B mesons are now being studied at B factories which collide electrons and positrons. These machines set the centre-of-mass energy of the colliding electrons and positrons to be equal to the mass of one of the Υ resonances. The $\Upsilon(4S) = 4^3S_1$ is where most of the B factories chose to run.

(ii) What is the quark content, the total spin (S) and the total angular momentum (J) of the $\Upsilon(4S)$? Why are these particular angular momentum states produced in electron-positron collisions?

[4 marks]

(iii) Why is it useful to set the beam energy equal to the mass of one of the Υ states when studying B mesons? Why is the $\Upsilon(4S)$ state chosen by B factories for studying B^0 mesons?

[3 marks]

The B_s^0 meson consists of an s and a $\overline{b}$ quark.

(iv) At which of the Υ resonances is it possible to observe the reaction $e^+e^- \to B_s^0\overline{B}_s^0$?

[3 marks]

(v) The B_s^0 can *mix* with the $\overline{B}_s^0$. This mixing takes place through a weak interaction involving the exchange of two W bosons called a box diagram. Draw a Feynman box diagram for B_s^0-$\overline{B}_s^0$ mixing. Be careful to label all the quarks and bosons. Describe why observing the reaction $e^+e^- \to B_s^0 B_s^0$ could be interpreted as evidence for B_s^0-$\overline{B}_s^0$ mixing.

[3 marks]

(vi) Two of the final states into which the B_s^0 can decay are $\psi\phi$ and $D_s^-\pi^+$. Draw a Feynman diagram of both of these decays of the B_s^0. Which of these decays could in principle be used to look for mixing of B_s mesons?

[3 marks]

[TOTAL 20 marks]

$B^0 = \overline{b}d$, $\overline{B}^0 = b\overline{d}$
$D_s^+ = c\overline{s}$, $D_s^- = \overline{c}s$
$B_s^0 = \overline{b}s$, $\overline{B}_s^0 = b\overline{s}$
$\psi = c\overline{c}$, $\phi = s\overline{s}$

$M_{B^0} = 5279\,\mathrm{MeV}/c^2$, $M_{B_s^0} = 5369\,\mathrm{MeV}/c^2$
Mass of $\Upsilon(3S) = 10335\,\mathrm{MeV}/c^2$, $\Upsilon(4S) = 10580\,\mathrm{MeV}/c^2$, $\Upsilon(5S) = 10860\,\mathrm{MeV}/c^2$

4. (i) $^{214}_{83}Bi$ is radioactive and decays by alpha decay. The alpha energy is 6.201 MeV. Write down the A and Z values of the nucleus left after the alpha decay, and calculate the difference in the nuclear masses of $^{214}_{83}Bi$ and this nucleus.

[7 marks]

(ii) $^{214}_{84}Po$ is also radioactive and decays by alpha decay. The alpha energy in this case is 7.829 MeV. If two samples containing equal amounts of either $^{214}_{83}Bi$ or $^{214}_{84}Po$ are mixed up, how could you identify which was which using just a particle counter (which **can not** measure the alpha energy). How clear cut would the result be? Justify your answer with numerical *estimates* where possible.

[7 marks]

(iii) In addition to the 6.201 MeV alpha particle from $^{214}_{83}Bi$ other alpha energies are also seen directly from this same nucleus. The energies of these are 6.161, 5.873, 5.728 and 5.709 MeV. How can this be explained given that the decays all involve the same starting and ending nuclei? Illustrate your answer with a diagram and suggest how you could verify your answer experimentally.

[6 marks]

[TOTAL 20 marks]

$m_p = 1.67262 \times 10^{-27}$ kg
$m_n = 1.67495 \times 10^{-27}$ kg
alpha particle $B_E = 28.3$ MeV
$c = 2.998 \times 10^{8}\ \mathrm{ms}^{-1}$
$e = 1.602 \times 10^{-19}$ C
$h = 6.626 \times 10^{-34}$ Js

5. The top quark was discovered at Fermilab where protons and anti-protons are made to collide head-on in an accelerator called the Tevatron.

(i) The centre-of-mass energy of the collisions at the Tevatron is 1.8 TeV. By design the centre-of-mass of the proton and anti-protons involved in the collision is at rest in the laboratory. What is the energy of the protons and anti-protons in the laboratory? [4 marks]

(ii) In a fixed-target experiment a beam of anti-protons is fired at a target of protons which are at rest. How much energy would an anti-proton need to have in order to achieve a centre-of-mass energy of 1.8 TeV in a collision with a proton which is at rest? Would the centre-of-mass for this collision be at rest in the lab frame? Comment briefly on why the Tevatron uses head-on collisions of protons and anti-protons. [4 marks]

At the Tevatron, top quarks are created when pairs of quarks in the proton and anti-proton collide in the reaction $q\overline{q} \to t\bar{t}$.

(iii) Draw a Feynman diagram of this reaction at the quark level. What force is involved in this interaction? Give a rough estimate of the total energy available to the $q\overline{q}$. [3 marks]

(iv) The top quark always decays via the reaction $t \to b + W$. The W can then decay into one of the final states indicated in the table below. Draw a Feynman diagram of the decay of the t quark where the W decays into an electron. Be careful to label the charge of all particles on the diagram. What force is involved in this interaction? [3 marks]

(v) Given the table of W decays given below, how likely is one to observe a final state involving a muon and an electron when a $t\bar{t}$ pair is created? [3 marks]

(vi) How often will the $t\bar{t}$ final state involve six quarks? What will be visible in a detector in these events? [3 marks]

[TOTAL 20 marks]

W Decay	Branching Fraction (%)
$e\nu_e$	10.6
$\mu\nu_\mu$	10.6
$\tau\nu_\tau$	10.6
$u\overline{d}$	34.1
$c\overline{s}$	34.1

6. (i) The gradient of the high-A linear part of the binding energy per nucleon curve is

$$\frac{\partial\left(\frac{B_E}{A}\right)}{\partial A} \sim -7.7 \times 10^{-3}\,\mathrm{MeV/A^2}.$$

Use this to estimate the amount of energy released when a uranium nucleus ($A = 238$) undergoes spontaneous fission into two equal mass fragments. [5 marks]

(ii) The spontaneous fission half-life for U is 10^{16} years. Calculate the amount of U needed to provide a power release of 1 MW *purely* through spontaneous fission. [5 marks]

(iii) Greater power levels can be achieved using the neutron capture reaction

$$n + {}^{235}U \rightarrow {}^{236}U \rightarrow {}^{147}La + {}^{87}Br + 2n.$$

Natural uranium has 98% ${}^{238}U$ and 2% ${}^{235}U$. If the total cross-section for high-energy neutron capture in both of these is 10 barns, how far would the average fast neutron travel before capture? If the cross-section for neutron induced fission in ${}^{235}U$ is 1 barn, what percentage of the neutrons captured actually cause fission? [5 marks]

(iv) Explain how the fission percentage can be dramatically increased for use in viable power stations. [5 marks]

[TOTAL 20 marks]

1 barn is $10^{-28}\,\mathrm{m}^2$
$e = 1.602 \times 10^{-19}\,\mathrm{C}$
$\rho_U = 18.9 \times 10^3\,\mathrm{kg.m^{-3}}$
$m_n = 1.67495 \times 10^{-27}\,\mathrm{kg}$

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

SEMICONDUCTOR DEVICE PHYSICS

For Third-and Fourth-Year Physics Students

Tuesday 27th May 2003: 10.00 to 12.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Using band edge diagrams explain what happens when a *pn-junction diode* is at *zero bias*, *forward bias* and *flat-band*.

[6 marks]

(ii) Sketch the current-voltage characteristics of a *pn-junction diode*, indicating all the different regimes. What occurs at large reverse bias?

[4 marks]

(iii) How do the current-voltage characteristics of a *Schottky junction diode* differ from that of the *pn-junction diode*?

[2 marks]

(iv) Sketch the variation of the dopant density, electric field, potential and band curvature in the depletion region of an *n-type Schottky junction diode*. Assuming the depletion region approximation, show that the depletion region width is given by:

$$W = \sqrt{\frac{2\varepsilon_s(\Phi_m - \Phi_s - V)}{qN_d}}$$

where Φ_m and Φ_s are the metal and semiconductor workfunctions respectively, V is the applied forward bias, ε_s is the permittivity of the semiconductor, N_d is the donor density and q is the charge on an electron.

[5 marks]

(v) What types of devices are based on the *pn* and *Schottky junction diodes*?

[3 marks]

[TOTAL 20 marks]

2. (i) Consider an n-type metal-oxide-semiconductor (MOS) structure. Assume that the metal and semiconductor workfunctions are equal so that at zero applied bias (short circuit) the device structure is *flatband*. Explain, using band edge diagrams, what is meant by *accumulation, depletion* and *inversion*.

[6 marks]

(ii) Sketch the source-drain current-voltage characteristics of an n-type *enhancement* mode MOSFET, indicating the linear and saturation regimes, and explain what occurs at pinch-off.

[4 marks]

(iii) In the *linear regime* the charge density (per unit area) at any given point in the channel from $x = 0$ to $x = L$ is given by:

$$Q_s(x) = -\frac{C_i}{a}(V_G - V_T - V_x)$$

while the resistance of a small element dx is given by:

$$dR = dx/Z\mu_n' Q_s(x)$$

where the symbols have their usual meaning (see list below for definition). By using Ohms law and integrating along the channel length, show that:

$$I_{DS} = \left(Z\mu_n' C_i/aL\right)\left((V_G - V_T)\,V_{DS} - V_{DS}^2/2\right).$$

[4 marks]

(iv) Sketch the *transfer characteristics*, indicating the regime covered by the answer to part (iii).

[2 marks]

(v) What factors determine the *threshold (turn-on) voltage* for *strong inversion*? Why would this be particularly large in a Gallium Arsenide metal-insulator-semiconductor FET device compared to a typical Silicon MOSFET?

[4 marks]

[TOTAL 20 marks]

$Q_s(x)$: carrier concentration per unit area in the channel at position x.
C_i : insulator capacitance.
a : Gate/channel area $= Z \times L$.
Z : Width of the gate/channel.
L : Length of the gate/channel.
V_G : Gate voltage.
V_T : Threshold voltage.
V_x : Voltage in the channel at position x.
μ_n' : average carrier mobility in the channel.
I_{DS} : Drain current.
V_{DS} : Drain voltage relative to source.

3. (i) For a 3 dimensional cube of sides length L, what are the Born-von Karman boundary conditions? Name two reasons why they are used in band theory. [4 marks]

(ii) In the free electron model all the possible $\underline{k}$ values can be placed in a 3D cubic lattice in $\underline{k}$-space. How is the volume occupied by each point in $\underline{k}$- space related to the length L in part (i)? [2 marks]

(iii) Let N electrons be placed in the 3D lattice. By considering the volume of the Fermi sphere, calculate the relationship between N and the Fermi wavevector k_F. Then, using the free electron model dispersion relationship, show that the Fermi energy E_F at the surface of the Fermi sphere is given by:

$$E_F = \frac{\hbar^2}{2m_e}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

where m_e is the free electron mass and $V = L^3$.

[4 marks]

(iv) What is the general definition of the density of states $D(E)$? Using the answer to part (iii), derive an expression for $D(E)$ for the free electron model. [3 marks]

(v) The dispersion curve of an electron of effective mass m_n^* in the conduction band of a semiconductor is:

$$E = E_C + \frac{\hbar^2 k^2}{2m_n^*}.$$

What is the density of states near the band edge? [2 marks]

(vi) What is the probability that a state in the conduction band at energy E will be occupied by an electron (assume $E - E_F \gg k_b T$)? Write down an integral expression for the number of electrons in the conduction band. [2 marks]

(vii) Sketch the variation with temperature of the carrier density in an n-type extrinsic semiconductor, giving brief explanations of the different regimes. Where would we want to operate a semiconductor device? Why? [3 marks]

[TOTAL 20 marks]

4. (i) Give a definition of the terms (a) optical absorption coefficient, α and (b) optical gain coefficient, g, in relation to the interaction between electromagnetic waves and a semiconductor.

[3 marks]

(ii) By considering propagation of an electromagnetic wave inside a semiconductor laser cavity with refractive index n and length l, show that the following relation is satisfied if laser oscillation is to persist

$$g \geq \alpha + \frac{2}{l} \ln \left(\frac{n+1}{n-1} \right)$$

[5 marks]

(iii) Sketch the structure of a generic in-plane double heterojunction semiconductor laser. Indicate the major components and list their functions in the operation of the laser.

[3 marks]

(iv) An in-plane GaAs semiconductor diode laser is operating at threshold condition. The laser cavity is formed by smooth parallel cleavage surfaces $3 \times 10^{-3} m$ apart and the absorption coefficient is $3500 m^{-1}$. Evaluate the minimum gain coefficient necessary for lasing action.

[3 marks]

(v) By suitable coating on *one* of the cleavage surface mirrors, its reflectivity is increased to nearly 100%. The gain of the semiconductor remains the same. Explain your reasoning and calculate how much shorter can one make the device while still maintaining the lasing action?

[3 marks]

(vi) Would the changes in the gain of the semiconductor brought about by an increase in forward bias modify the axial mode frequency of the cavity defined by $\nu = \frac{mc}{2nl}$? Here m is an integer, c is the speed of light in vacuum, n is the real part of refractive index of the semiconductor and l is the cavity length.

[3 marks]

[TOTAL 20 marks]

The electric field reflection coefficient at the air-semiconductor interface for normal incidence is given by $\frac{1-n}{1+n}$.

The relative dielectric constant of GaAs is 10.8.

5. (i) Describe what is meant by the term *photovoltaics* as applied to a pn junction under illumination.

[4 marks]

(ii) Sketch the I-V characteristics of an ideal pn junction solar cell in the *dark* and under *illumination*. Explain why, under normal operating conditions, the current flow due to optically induced current is negative in relation to the voltage appearing on the terminals of the diode.

[5 marks]

(iii) Sketch an equivalent circuit of an ideal pn junction solar cell operating under the photovoltaic mode.

From the equivalent circuit, or otherwise, derive an expression for the current flow through the external load as a function of voltage V across the diode and optically induced current I_{op}.

[5 marks]

(iv) Show that the open circuit voltage of this ideal device is given by:

$$V_{OC} = \frac{k_B T}{e} \ln \left[\frac{I_{op}}{I_0} - 1 \right].$$

[2 marks]

(v) It appears from the result of the previous section that V_{OC} can be increased by increasing illumination and I_{op}. Explain, with the aid of band edge diagram, that the open circuit voltage can never exceed the band gap of the semiconductor material measured in volts.

[4 marks]

[TOTAL 20 marks]

6. Make short notes (and sketches where appropriate) on FOUR of the following topics.

(i) Outline practical methods to maximise both the efficiency and the speed of bipolar junction transistor.

[5 marks]

(ii) The use of Silicon and III-V compound semiconductors in different devices.

[5 marks]

(iii) The different zone schemes in the description of dispersion relations in crystalline solid.

[5 marks]

(iv) Advantages of a two dimensional semiconductor structure, such as multi-quantum-wells, over the three dimensional counterpart in light emitters.

[5 marks]

(v) Issues in choosing semiconductor materials in photodetectors for optical communication applications.

[5 marks]

(vi) Use of *inter-subband transitions* in quantum cascade lasers.

[5 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ATMOSPHERIC PHYSICS

For Third - and Fourth -Year Physics Students

Tuesday 3rd June 2003: 14.00 to 16.00

Answer ALL parts of Section A and TWO questions from Section B.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A (Compulsory)

Use the YELLOW answer book when attempting questions from this Section.

1. Answer all parts of this question. You should complete the statements noting that each part has only one correct answer.

Marks will NOT be awarded for any answer containing more than one letter.

Any rough workings must be done in the space provided in the YELLOW answer book.

(M1) An air parcel undergoing an adiabatic change will

- (a) always have a lower pressure,
- (b) always have a higher temperature,
- (c) have a constant potential temperature,
- (d) undergo latent heating,
- (e) have increased it's entropy.

(M2) The change of temperature with height

- (a) can be used to infer the stability of a parcel of air,
- (b) is independent of latitude,
- (c) is independent of longitude,
- (d) increases with latent heating,
- (e) is $-6\,$K per km in a dry atmosphere.

(M3) Specific humidity

- (a) equals to relative humidity divided by the saturation vapour pressure,
- (b) generally decreases with height,
- (c) is reported in "%",
- (d) is not important for the natural greenhouse effect,
- (e) has a small seasonal variation over the UK.

(M4) Latent heat

- (a) is released when cloud droplets form,
- (b) does not alter the temperature structure of the atmosphere,
- (c) is inversely proportional to the relative humidity,
- (d) is released when ice evaporates,
- (e) decreases the potential temperature.

(M5) When the buoyancy force is in the same direction as the displacement of a parcel then

- (a) the environmental lapse rate equals that of a parcel of air,
- (b) the atmospheric layer is said to be unstable,
- (c) the amount of moisture can be ignored,
- (d) the temperature of the parcel of air is smaller than the environmental temperature,
- (e) clouds are unlikely to form.

(M6) The flux density is defined as

(a) the power per unit area per solid angle,
(b) the irradiance per unit area per unit frequency interval,
(c) the power per unit solid angle,
(d) the power per unit area normal to the surface,
(e) the radiance per unit frequency interval.

(M7) The tropospheric optical depth (tropopause is at 10 km) of a scatterer with cross-section of $10^{-20}\,\mathrm{m}^2\,\mathrm{kg}^{-1}$ and a mean density of $10^{18}\,\mathrm{g\,m}^{-3)}$ is about

(a) 10^{-5},
(b) 10^{5},
(c) 1,
(d) 100,
(e) 0.1.

(M8) Clouds

(a) have a larger albedo than the oceans,
(b) always cool the climate system,
(c) have a smaller albedo than the surface of ice sheets,
(d) are strong absorbers of incoming solar radiation,
(e) are transparent in the infra-red.

(M9) Carbon dioxide

(a) is the most important “greenhouse” gas,
(b) has a mass mixing ratio which increases with height,
(c) reacts with ozone to create the ozone hole,
(d) absorbs nearly all incoming short wave radiation,
(e) increases the infra-red optical depth of the atmosphere.

(M10) The net radiative heating rate

(a) is independent of solar heating,
(b) proportional to the difference of the downward minus the upward fluxes,
(c) mostly positive in the troposphere,
(d) mostly negative in the stratosphere,
(e) is a maximum at the tropopause.

(M11) Relative vorticity

(a) is smallest near the equator,
(b) is proportional to the acceleration of the flow,
(c) is always conserved,
(d) is defined as the curl of the velocity vector,
(e) is proportional to the lapse rate.

(M12) Anticyclonic systems in the Northern Hemisphere

(a) have anticlockwise flow in the lower troposphere,
(b) enhance cloud cover,
(c) move fast,
(d) are divergent near the surface,
(e) have a core of low pressure.

(M13) The thermal wind :

(a) is due to the balance of the pressure gradient with the Coriolis force,
(b) also known as curvilinear wind,
(c) is the vertical wind shear due to the horizontal temperature gradient,
(d) is proportional to the vertical temperature gradient,
(e) is due to balance of the centrifugal force and the pressure gradient.

(M14) Rossby waves

(a) reduce vertical motion,
(b) have gravity as their restoring force,
(c) horizontally propagate relatively against the mean flow,
(d) have typical wavelengths of less than about 10 km,
(e) have typical periods of minutes.

(M15) For a pressure difference of 1 h Pa over 100 km the geostrophic wind speed at 50°N (assuming the air density of $1\,\mathrm{kg\,m^{-3}}$) is about

(a) $1\,\mathrm{m\,s^{-1}}$,
(b) $20\,\mathrm{m\,s^{-1}}$,
(c) $150\,\mathrm{m\,s^{-1}}$,
(d) $10\,\mathrm{m\,s^{-1}}$,
(e) $3\,\mathrm{m\,s^{-1}}$.

SECTION B

2. (i) What is the dry and saturated adiabatic lapse rate and what assumptions are made in their derivations.

[5 marks]

(ii) Show that the dry adiabatic lapse rate equals the acceleration due to gravity divided by the heat capacity of dry air.

[5 marks]

(iii) A parcel of air has an air temperature of 15° C near the ground and a dew point temperature of 8° C. What would the environmental temperature at 0.5 km need to be for the air parcel to be considered unstable? At what height would saturation occur and a cloud base form? What would the approximate temperature at cloud top of this 1 km high cloud be? How could fog form?

[10 marks]

[TOTAL 20 marks]

3. (i) From the definition of optical depth, χ, and emission intensity, I, derive the radiative-transfer equation:

$$\frac{dI}{d\chi} + I = B \tag{3.1}$$

where B is the Planck function.

[10 marks]

(ii) What is the physical interpretation of equation 3.1 in terms of the radiance at any given height in the atmosphere?

[5 marks]

(iii) How can equation 3.1 be used to calculate the net radiative heating rates in the atmosphere?

[5 marks]

[TOTAL 20 marks]

4. (i) What is vorticity and why is it useful for understanding the weather?

[5 marks]

(ii) Consider a long-wave pattern at the level of non-divergence in a zonal current of uniform constant velocity, U. From the principal of the conservation of absolute vorticity and making the assumption that the total velocities are independent of latitude show the Rossby wave equation can be written as:

$$(U - c)\frac{\delta^2 v'}{\delta x^2} + v'\beta = 0$$

where c is the wave velocity, v' is the meridional velocity perturbation and β is the meridional rate of change of the Coriolis parameter. Find an expression for c in terms of the wavelength and β.

[10 marks]

(iii) Calculate the wavelength of the Rossby wave for appears stationary at 50°N when the zonal wind is $50\,\mathrm{m\,s^{-1}}$. For the same zonal wind, what would happen to a wave of the same wavelength at lower latitudes?
(The radius of the Earth is 6400 km.)

[5 marks]

[TOTAL 20 marks]

5. Write notes on ONE of the following:

(i) The geostrophic wind.

(ii) Clouds.

(iii) The greenhouse effect.

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

BIOPHYSICS OF NERVE CELLS & NETWORKS

For Third-and Fourth-Year Physics Students

Monday 19th May 2003: 14.00 to 16.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

$R = 8.314\ \mathrm{JK^{-1}mol^{-1}}$, $F = 96,400\ \mathrm{C\,mol^{-1}}$
The distributions of sodium, potassium and chloride ions across the membrane of a nerve cell can be taken to be:

Ion	**Inside**	**Outside**
Sodium	14 mM	125 mM
Potassium	124 mM	5 mM
Chloride	6 mM	77 mM

1. (i) Starting from the relationship between the membrane current per unit length and the variation of membrane potential along an unmyelinated nerve axon, show that the current i_m that flows per unit area across the membrane is given by

$$i_m = \frac{a}{2\rho_i\theta^2}\left(\frac{\partial^2 V_m}{\partial t^2}\right)$$

where ρ_i is the resistivity of the axoplasm, a is the axon radius, θ is the velocity of the action potential and V_m is the membrane potential. [12 marks]

(ii) What can be inferred from the equation in part (i) about how the velocity of the action potential in an unmyelinated nerve varies with axon diameter? [3 marks]

(iii) What is meant by the “refractory period” and how does it affect the properties of the nerve axon? [2 marks]

(iv) What is myelin and how does it affect the propagation of nerve action potentials? [3 marks]

[TOTAL 20 marks]

2. (i) Explain the principle behind the voltage-clamp method for studying the electrical properties of nerve membranes.

[4 marks]

(ii) How was this method employed to describe the time-course of the changes in sodium ion conductance in the membrane of the squid giant axon following a sudden depolarisation?

[6 marks]

(iii) Hodgkin and Huxley described the sodium current across the axonal membrane in terms of the following equation

$$I_{Na} = m^3 h \bar{g}_{Na} \left(V_m - V_{Na}\right).$$

What do the various parameters in this equation represent and what are the main assumptions made in its derivation?

Show that the time course of h is given by an equation of the form:

$$h = h_\infty - (h_\infty - h_0) \exp\left(\frac{-t}{\tau_h}\right).$$

[6 marks]

(iv) Sketch the time-course of the parameter h under the following conditions

(a) The axon has been held under voltage clamp for a prolonged period at -100 mV and is then stepped to 0 mV.

[2 marks]

(b) The axon has been held under voltage clamp for a prolonged period at 0 mV and is then stepped to -100 mV.

[2 marks]

[TOTAL 20 marks]

3. (i) From first principles, show that the difference in electrochemical potential across a membrane can be written

$$\Delta\mu = RT \ln\left(\frac{c_{in}}{c_{out}}\right) + zFV_{mem} \,.$$

[7 marks]

(ii) From this equation derive the Nernst equation and show, using the ionic concentration data given in the Table, that a nerve cell cannot be at thermodynamic equilibrium. [5 marks]

(iii) Consider a neuron under voltage-clamp held at a membrane potential of $-40\,\mathrm{mV}$. Assuming the cell expressed both $\mathrm{GABA_A}$ receptors and nicotinic acetylcholine receptors, would inward or outward currents be observed when each of these neurotransmitters was applied to the cell? Explain your reasoning.

[4 marks]

(iv) In a voltage clamp experiment, how would you determine if these currents would be excitatory or inhibitory in the un-clamped neuron? [6 marks]

[TOTAL 20 marks]

4. (i) Describe the basic mechanisms underlying the transmission of information across a chemical synapse. How do chemical synapses differ from electrical synapses?

[5 marks]

(ii) Describe two experiments that would show that calcium must move across the presynaptic membrane in order to trigger neurotransmitter release.

[5 marks]

(iii) What evidence suggests that transmitter release is quantal at the neuromuscular junction?

[5 marks]

(iv) Outline how the patch clamp technique can be used to study the properties of postsynaptic neurotransmitter receptors. Under what circumstances might noise analysis be used instead of patch-clamping?

[5 marks]

[TOTAL 20 marks]

5. Are the following statements true or false? Explain your reasoning (N.B. no reasoning, no marks).

(i) The height of an action potential is affected by external sodium ion concentrations but this has little effect on the resting membrane potential. [2 marks]

(ii) For myelinated nerves, the ratio of the internal to the external diameter is roughly constant. [2 marks]

(iii) Ion flow down an open voltage-gated sodium channel does not require energy. [2 marks]

(iv) Sustained action potentials result in the consumption of energy. [2 marks]

(v) The breakdown field for nerve membranes is $50,000\,\mathrm{V\,cm^{-1}}$. [2 marks]

(vi) The time-course of an excitatory postsynaptic current is determined by the diffusion of the neurotransmitter away from the postsynaptic membrane. [2 marks]

(vii) A positively charged local anaesthetic such as lidocaine binds tighter to its target at depolarised potentials. [2 marks]

(viii) Rod cells in the retina depolarise when exposed to light. [2 marks]

(ix) Visual acuity at night (as opposed to during the day) is enhanced at the expense of sensitivity. [2 marks]

(x) Colour blindness is usually a consequence of a vitamin deficiency. [2 marks]

[TOTAL 20 marks]

6. Write notes on ALL THREE of the following:

(i) The measurement of the Hodgkin-Huxley parameters h_∞ and τ_h

(ii) The role of the sodium pump in neuronal excitability.

(iii) The role of horizontal cells in contrast enhancement in the retina.

[20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

CONDENSED MATTER THEORY

For Third- and Fourth-Year Physics Students

Wednesday 21st May 2003: 14.00 to 16.00

Answer ONE question from Section A, ONE question from Section B and ONE question from Section C.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. Consider the percolation theory problem in a lattice of infinite size $L = \infty$ with critical occupation probability p_c.

(i) (a) Define the order parameter $P(p, L = \infty)$ for the geometrical phase transition. [2 marks]

(b) Describe the behaviour of the order parameter as a function of p. Illustrate your explanation with a sketch. [2 marks]

(c) Let $n_s(p, L = \infty)$ denote the number of s-clusters per lattice site. Justify the relation

$$P(p, L = \infty) = p - \sum_{s=1}^{\infty} s n_s(p, L = \infty). \qquad (1.1)$$

[2 marks]

In the following $p > p_c$. Let $\xi(p)$ denote the correlation length and assume that the order parameter becoming non-zero for p approaching p_c from above is characterised by the critical exponent β, that is,

$$P(p, L = \infty) \propto (p - p_c)^{\beta} \qquad \text{for } p \to p_c^+.$$

(ii) (a) Show that for $p \to p_c^+$, $P(\xi, L = \infty) \propto \xi^{-\beta/\nu}$, where ν is the critical exponent characterising the divergence of the correlation length as $p \to p_c$. [2 marks]

(b) Argue why, for finite lattices $L < \infty$, with $L \ll \xi$ one would expect

$$P(\xi, L) \propto L^{-\beta/\nu}. \qquad (1.2)$$

[2 marks]

(c) Numerically, how would you determine the ratio $-\beta/\nu$? [2 marks]

(iii) In finite lattices, $L < \infty$, you may assume the cluster number distribution at $p = p_c$ obeys the scaling law (for all s)

$$n_s(p_c, L) = s^{-\tau} g(s/L^D), \qquad (1.3)$$

where τ is the cluster number exponent, D is the fractal dimension and the scaling function $g(x)$ is constant for $x \ll 1$ and decays rapidly for $x \gg 1$. In addition, you may assume that

$$P(p_c, L) = \sum_{s=1}^{\infty} s\, s^{-\tau} g(0) - \sum_{s=1}^{\infty} s n_s(p_c, L). \qquad (1.4)$$

(a) Show that Eq.(1.4) is correct in the limit of $L \to \infty$. [2 marks]

(b) Combining Eq.(1.2), Eq.(1.3) and Eq.(1.4) or otherwise, derive the scaling relation

$$-\beta/\nu = D(2 - \tau).$$

[6 marks]

[TOTAL 20 marks]

2. Consider site percolation on the square lattice in two dimensions and let p denote the occupation probability.

(i) (a) How many different microstates are associated with a 2×2 lattice? [2 marks]

(b) Sketch all the microstates and list the probability of each. [2 marks]

(ii) (a) Using cells of size $b \times b$ and adopting the rule of having a *vertically* spanning cluster to define a renormalisation group transformation $R_b(p)$, show that

$$R_b(p) = 2p^2 - p^4, \tag{2.1}$$

when $b = 2$. [2 marks]

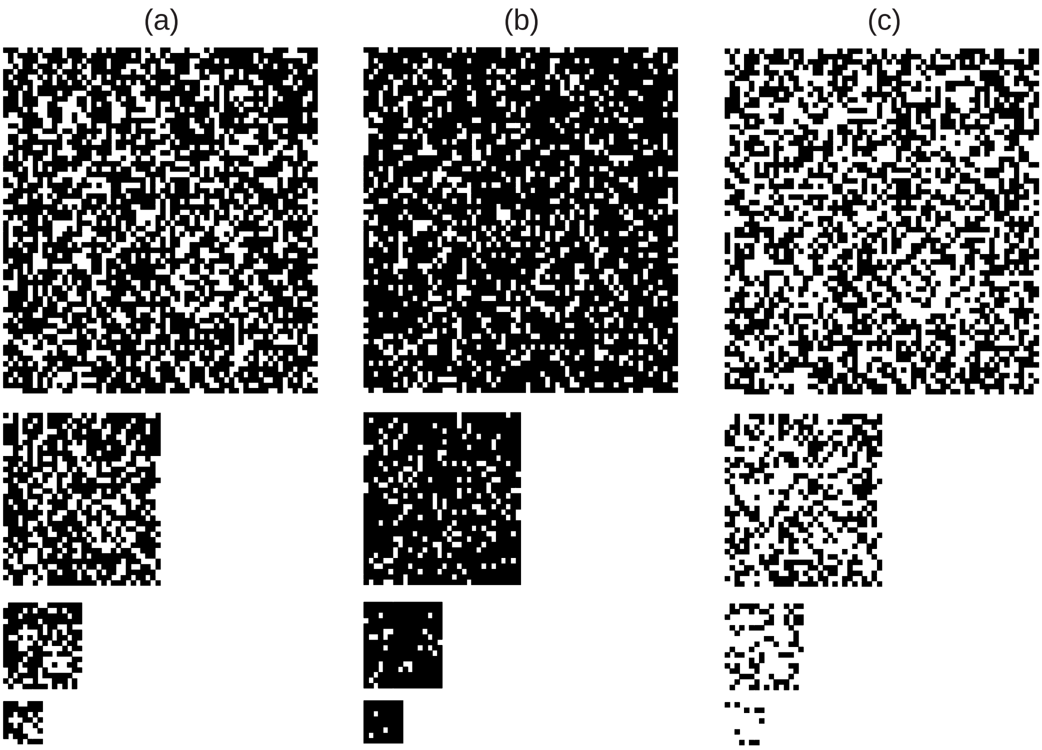
(b) Sketch the graph of the renormalisation group transformation in Eq.(2.1) and identify clearly the fixed points $p^\star$. [2 marks]

(c) Identify the critical occupation probability p_c and determine the correlation length exponent ν predicted by the renormalisation group transformation. [4 marks]

(iii) In the diagrams below, lattices of size $L \times L = 64 \times 64$ with different initial occupation probabilities p_a, p_b, and p_c have been renormalised ($L \to L/2 \to L/4 \to L/8$) using the renormalisation group transformation Eq.(2.1).

(a) Discuss the sequences of renormalised lattices in diagrams (a), (b) and (c) in terms of the flow in p-space. [5 marks]

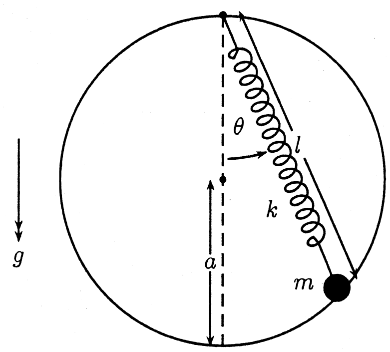
(b) Relate the concept of fixed points for a renormalisation group transformation to scale invariance. [3 marks]



[TOTAL 20 marks]

SECTION B

3. A bead of mass m is suspended from a spring with spring constant k such that it rests inside a circular ring of radius a, which lies in the vertical plane. The acceleration due to gravity is g and you may assume there is no friction. Under the weight of the bead the spring extends from its natural length a to a length l. The spring makes an angle θ to be measured positive counterclockwise from the vertical. Geometry reveals that $l = 2a\cos\theta$.



(i) Show that the total energy of the mass-spring system is

$$U(\theta) = \frac{1}{2}ka^2(2\cos\theta - 1)^2 - mga\cos 2\theta, \tag{3.1}$$

where the zero of the gravitational potential energy is defined at the horizontal passing through the centre of the circle. [2 marks]

(ii) (a) Show that the system is in equilibrium when

$$\theta = 0 \text{ or } \theta = \pm\arccos\left(\frac{ka}{2(ka - mg)}\right).$$

[2 marks]

(b) Show that $ka \geq 2mg$ for the existence of a non-trivial equilibrium angle θ. [2 marks]

(iii) Expanding the function $U(\theta)$ in Eq.(3.1) around $\theta = 0$ up to fourth order, we find that

$$U(\theta) = \left(\frac{1}{2}ka^2 - mga\right) + a(2mg - ka)\theta^2 + \frac{a}{12}(7ka - 8mg)\theta^4. \tag{3.2}$$

(a) Explain why only terms of even order appear in the expansion, Eq.(3.2). [2 marks]

(b) Sketch the function $U(\theta) - \left(\frac{1}{2}ka^2 - mga\right)$ for $2mg > ka$ and $2mg < ka$ assuming the coefficient of θ^4 in Eq.(3.2) is positive. [2 marks]

(c) Sketch the physical solution θ as a function of the ratio $\dfrac{2mg}{ka}$. Relate the graph to the sketch from (iii)(b). [2 marks]

(d) Briefly outline the Landau theory of second-order phase transitions for the ferromagnetic spin 1/2 Ising model. [6 marks]

(e) What is the order parameter of the mass-spring system? Explain your answer. [2 marks]

[TOTAL 20 marks]

4. (i) The order parameter for a superconducting phase transition is a complex number

$$\Psi(\mathbf{r}) = |\Psi(\mathbf{r})| \exp(i\Theta(\mathbf{r})). \tag{4.1}$$

(a) What is the physical interpretation of $|\Psi(\mathbf{r})|^2$? [2 marks]

(b) Explain why one can assume that $|\Psi(\mathbf{r})|^2$ does not change significantly in space, that is, we can write $|\Psi(\mathbf{r})|^2 = |\Psi|^2$. [2 marks]

(ii) You may assume without proof the equation of continuity for probabilities

$$\mathbf{j}_{prob} = \frac{1}{2m}[\Psi^\star(\mathbf{r})(-i\hbar\nabla - q_s\mathbf{A})\Psi(\mathbf{r}) + \Psi(\mathbf{r})(i\hbar\nabla - q_s\mathbf{A})\Psi^\star(\mathbf{r})]. \tag{4.2}$$

(a) Identify clearly all the terms in the equation of continuity and explain why the right hand side is a real number. [3 marks]

(b) Derive the equation for the electrical current density

$$\mathbf{j} = \frac{q_s}{m}[\hbar\nabla\Theta(\mathbf{r}) - q_s\mathbf{A}]|\Psi|^2. \tag{4.3}$$

[3 marks]

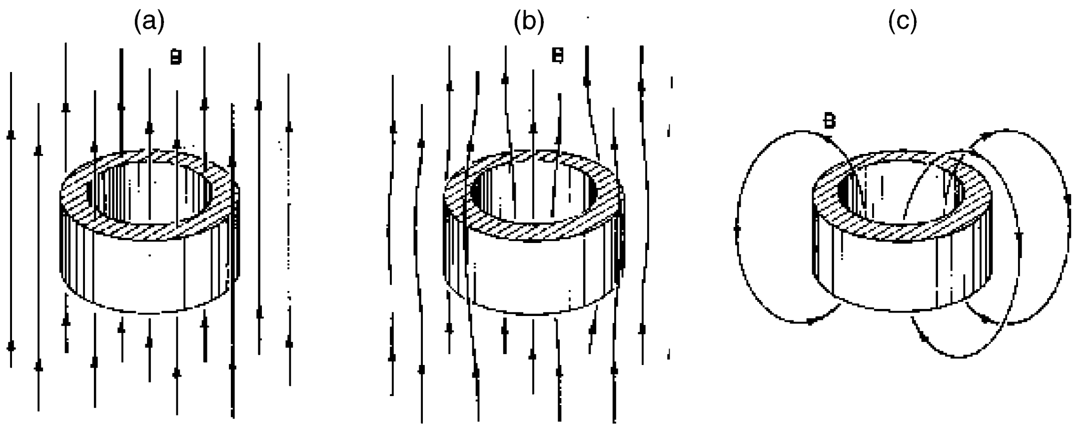
(c) What is the classical equivalent of Eq.(4.3)? [2 marks]

(iii) Consider a superconducting ring.

(a) Describe the phenomenon of flux quantisation. [2 marks]

(b) Show that Eq.(4.3) for the electrical current density implies flux quantisation. [2 marks]

(iv) Consider the diagrams below which display a superconducting ring in an external magnetic field at temperature (a) $T > T_c$ and (b) $T < T_c$. In (c), the external magnetic field has been switched off. Explain what happens when going from (a) to (b) and from (b) to (c). Indicate whether you will observe flux quantisation in the settings (a), (b) and (c). What is the source of the magnetic loops in diagram (c)? [4 marks]



[TOTAL 20 marks]

SECTION C

5. (i) (a) Write down the Hamiltonian $\mathcal{H}$ for a ferromagnetic spin 1/2 Ising model with exchange parameter J and external magnetic field of strength H. Clearly identify all the symbols. [2 marks]

(b) Discuss the simplifications entering into the Ising model. [2 marks]

(c) Argue why such a simple model might be relevant for a real physical ferromagnet near the critical temperature T_c. [2 marks]

In the following, assume zero external magnetic field $H = 0$.

(ii) (a) Define the order parameter for the Ising model. [2 marks]

(b) Discuss the microscopic states and the associated value of the order parameter in the limits $T \to 0$ and $T \to \infty$. [4 marks]

(c) Describe the behaviour of the order parameter as a function of T. Illustrate your explanation with a sketch. Relate the sketch to your answer in (ii)(b). Discuss qualitatively the microscopic states associated with $T = T_c$. [4 marks]

(iii) Given that the free energy at temperature T is

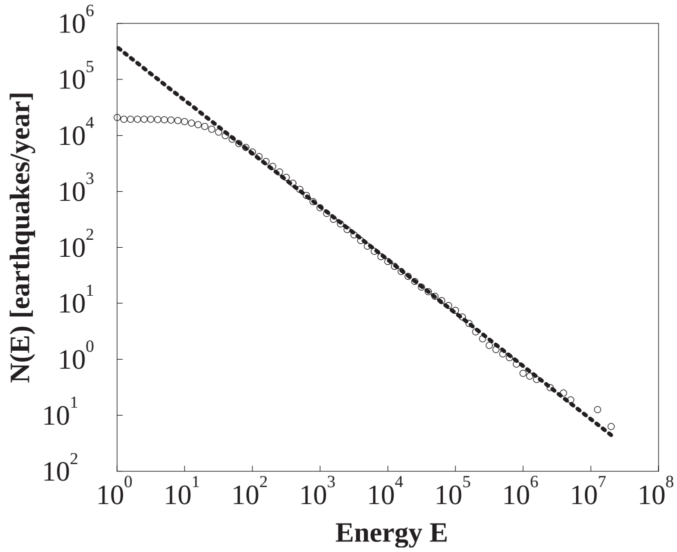
$$\mathcal{F} = U - T\mathcal{S}, \tag{5.1}$$

where U is the internal energy and $\mathcal{S}$ the entropy, explain why you would expect the Ising model to exhibit a phase transition. [4 marks]

[TOTAL 20 marks]

6. (i) The figure below displays the number of earthquakes $N(E)$ with energy release larger than E per year. Explain why this indicates that the seismic system might be viewed as being self-organised critical.

[2 marks]



(ii) In the Ising model at the critical temperature $T = T_c$, the susceptibility $\chi_T = \left(\frac{\partial\langle M\rangle}{\partial H}\right)_T$ diverges. Contrast this system with the metaphor of a slowly driven sandpile, explaining briefly which sandpile quantity is the analogue of the susceptibility, and the difference between criticality in equilibrium systems and non-equilibrium systems displaying scale invariance.

[4 marks]

(iii) In a model system displaying self-organised criticality, the avalanche size distribution obeys (for all avalanche sizes s)

$$P(s, L) = s^{-\tau} f(s/L^D), \tag{6.1}$$

where τ and D are critical exponents, L is the system size and the scaling function $f(x)$ is constant (different from zero) for $x \ll 1$ and decays rapidly for $x \gg 1$.

(a) Assume $L = \infty$. Given that $P(s, L = \infty)$ can be normalised but that the average response $\langle s\rangle = \sum_{s=1}^{\infty} sP(s, L = \infty)$ diverges, derive lower and upper bounds for the critical exponent τ.

[4 marks]

(b) Assuming L is finite, derive the scaling of the 4^{th} moment $\langle s^4\rangle = \sum_{s=1}^{\infty} s^4 P(s, L)$ with system size L in terms of τ and D.

[4 marks]

(c) Numerically, how would you determine the graph of the scaling function f for the model system?

[4 marks]

(iv) Why does it seem implausible that the scale invariance observed in equilibrium systems at a phase transition is related to the scale invariance observed in nature?

[2 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

COSMOLOGY

For Third-and Fourth-Year Physics Students

Wednesday 28th May 2003: 14.00 to 16.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

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Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) By applying the adiabatic form of the first law of thermodynamics, $\mathrm{d}E = -p\mathrm{d}V$, to an element of the substratum, derive the fluid equation. [3 marks]

(ii) In a Universe dominated by radiation the pressure is given by $p_r = \rho_r c^2/3$. Show that the solution to the fluid equation is:

$$\rho(t) = \rho_0 \left(R(t)/R_0\right)^{-4}$$

where ρ_0 and R_0 are the density and scale factor at the present time. [4 marks]

(iii) What is the corresponding solution for a matter dominated universe? [3 marks]

(iv) The Friedmann equations (for $\Lambda = 0$) are:

$$\begin{aligned}\ddot{R} &= -\frac{4\pi G\rho}{3}R \\ \dot{R}^2 &= \frac{8\pi G\rho}{3}R^2 - kc^2.\end{aligned}$$

Show from these and the fluid equation that the expansion of the early Universe is characterized by the expression $R(t) \propto t^{1/2}$. [4 marks]

(v) Assuming $k = 0$, the corresponding solution for a matter dominated Universe is $R(t) \propto t^{2/3}$. Estimate from this the time when matter and radiation were equally important constituents of the universe. [6 marks]

[TOTAL 20 marks]

[Assume the ratio of the present densities of radiation and matter to be $\rho_{r,0}/\rho_{m,0} = 3.5 \times 10^{-5}$ and the age of the Universe $t_0 = 14\,\mathrm{Gyr}$]

2. (i) The intensity of blackbody radiation is given by the Planck spectrum:

$$I(\nu)d\nu = B_\nu(T)d\nu = \left(\frac{2h}{c^2}\right)\left(\frac{\nu^3}{e^{h\nu/k_BT} - 1}\right)d\nu \qquad (2.1)$$

Where ν is the frequency, h is Planck's constant, k_B is Boltzmann's constant, and T is the temperature. Show that the total radiation density is given by the expression $u_{rad} = aT^4$, where a is a constant.

[4 marks]

(ii) The Cosmic Microwave Background (CMB) has $T = 2.73\,\mathrm{K}$. Evaluate the density parameter of the CMB, Ω_{rad}, at the present epoch.

[4 marks]

(iii) Show from equation 2.1 that the CMB retains its blackbody form during the expansion of the universe, and estimate the temperature and energy density at the time of recombination at redshift $z \sim 1000$.

[4 marks]

(iv) Observations of CMB fluctuations suggest that the current total density parameter of the Universe Ω_{tot} is very close to 1. Discuss the major contributors to the density of the Universe (in addition to radiation) and their current importance.

[8 marks]

[TOTAL 20 marks]

[You may use the following:
Useful formula $\displaystyle\int_0^\infty \frac{y^3}{e^y - 1}dy = \frac{\pi^4}{15}$
Gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-1}}$
Hubble constant $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$
$1\,\mathrm{pc} = 3.09 \times 10^{16}\,\mathrm{m}$
Speed of light $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$
Boltzmann constant $k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$]

3. (i) Discuss the main lines of evidence supporting the hot big bang theory. Illustrate your answer with equations or diagrams, as necessary.

[9 marks]

(ii) For an Einstein de Sitter model ($k = 0$, $\Lambda = 0$, $p = 0$), determine the age of the Universe, t_0, in years for a Hubble constant $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and discuss the consistency of this with other estimates.

[4 marks]

(iii) Observations suggest our own Universe has $k = 0$, and that the density parameter associated with the cosmological constant, $\Omega_\Lambda = 0.7$. Using the Friedmann equation (5.1), show that the age of the universe is given by:

$$t_0 = \int_0^{t_0} dt = \frac{1}{H_0}\int_0^1 \left(\frac{1-\Omega_\Lambda}{x} + \Omega_\Lambda x^2\right)^{-1/2} dx, \text{ where } x = \frac{R}{R_0}\,.$$

This has the solution:

$$t_0 = \frac{2}{3H_0}\Omega_\Lambda^{-1/2}\ \ln\left(\frac{1+\Omega_\Lambda^{1/2}}{(1-\Omega_\Lambda)^{1/2}}\right)\,.$$

For the Hubble constant above, evaluate the age of the Universe in this case and comment also on the consistency of this result.

[7 marks]

[TOTAL 20 marks]

[$1\ \mathrm{pc} = 3.09 \times 10^{16}\,\mathrm{m}$]

4. (i) Explain what is meant by cosmological inflation, and describe how this may solve three problems associated with the standard big bang model. [7 marks]

(ii) Some cosmological models predict that the Universe expands as $R \propto t^m$ where m is an arbitrary constant. What range of values of m corresponds to a satisfactory inflationary expansion? [4 marks]

(iii) Magnetic monopoles behave as non-relativistic matter. Suppose that at a temperature corresponding to the Grand Unified era, about 3×10^{28} K, magnetic monopoles were created with a density of $\Omega_{mon} = 10^{-10}$. Assuming that the Universe has a critical density and is radiation dominated, what would the temperature be when the density of monopoles equals that of radiation? [4 marks]

(iv) In the present Universe, $T \approx 3$ K. Compute the value $\Omega_{mon}/\Omega_{rad}$ would have at the present day. Assuming $\Omega_{rad} = 10^{-4}$, is this ratio compatible with observations? If the radiation density stays constant during inflation, how much inflationary expansion is necessary so that the present-day density of monopoles matches that of radiation? [5 marks]

[TOTAL 20 marks]

5. (i) In General Relativity, the Friedmann equation is:

$$\dot{R}^2 = \frac{8\pi G\rho}{3}R^2 - kc^2 + \frac{\Lambda}{3}R^2 \tag{5.1}$$

where R is the scale factor of the universe, G is the gravitational constant, ρ is the density of matter, c is the speed of light and k is a constant. For $\Lambda = 0$, discuss the solution, $R(t)$, of the Friedmann equation in a matter-dominated universe for the following cases:

(a) Where the mass of matter in the Universe is negligible

(b) Where the mass is non-negligible and $k = 0$

(c) Where the mass is non-negligible and $k = -1$

(d) Where the mass is non-negligible and k = 1

Illustrate your answers with a graph of $R(t)$ in each case. [10 marks]

(ii) What is the meaning of the term Λ in Eq. 5.1? Give examples of the interpretation of this term and briefly describe the evidence that, at the current epoch, $\Lambda > 0$. [4 marks]

(iii) For a universe with $\Lambda > 0$ and $k \leq 0$, describe the behaviour of the scale factor $R(t)$ and derive the late-time behaviour of $R(t)$ in this case. Discuss the significance for our own Universe.

[6 marks]

[TOTAL 20 marks]

6. (i) What is the meaning of the term k in the Robertson-Walker Metric:

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right).$$

State why k can take values of -1, 0 and $+1$ without losing generality, and describe Universes characterized by these values.

[4 marks]

(ii) In an expanding universe described by the Robertson-Walker metric, by considering photons moving radially towards us from a distant observer, show that the light signal is redshifted by a factor:

$$1 + z = \frac{R_{t_0}}{R_{t_e}}$$

where t_0 and t_e are the times at which the photon is observed and emitted respectively.

[6 marks]

(iii) The most distant object currently known is a quasar at $z = 6.43$. Assuming a matter-dominated Universe with $k = 0$,

(a) How much more dense was the universe at this epoch, compared to the present one, and what was its temperature?

[2 marks]

(b) What is the proper distance to the quasar?

[5 marks]

(iv) What observational and theoretical factors limit our ability to see objects at even greater distances?

[3 marks]

[TOTAL 20 marks]

[Hubble constant $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$
$1\mathrm{pc} = 3.09 \times 10^{16}\,\mathrm{m}$]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

DYNAMICAL SYSTEMS & CHAOS

For Third - and Fourth - Year Physics Students

Tuesday 27th May 2003: 14.00 to 16.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. The members of two similar species compete with members of their own and of the other species for food and living space. The populations $x(t)$, $y(t)$ satisfy as functions of time t the equations

$$\begin{aligned}\frac{dx}{dt} &= x\left(\frac{1}{2} - \frac{3x}{2} - \frac{1}{4}y\right) \\ \frac{dy}{dt} &= y(1 - y - 2x) .\end{aligned}$$

Determine the critical points and their stability in the linear approximation.

Find also the local eigenvectors and use them to sketch the phase portrait of the system in all four quadrants. Hence show that almost all initial populations $x(0)$, $y(0)$ tend eventually to a state of coexistence corresponding to an asymptotically stable node.

[20 marks]

2. The simple S, I, R model equations for the transmission of a disease are

$$\begin{aligned} \frac{dS}{dt} &= -aSI \\ \frac{dI}{dt} &= aSI - bI \\ \frac{dR}{dt} &= bI \end{aligned}$$

where $S(t)$, $I(t)$, $R(t)$ are respectively susceptibles, infectives, removed/recovered and a, b are positive constants.

(i) Show that the overall population $N = S + I + R$ remains constant, so that we may consider (S, I) in a projected phase plane. Hence show that a trajectory with initial values (S_0, I_0) has equation $I(S) = I_0 + S_0 - S + \frac{b}{a} \ln\left(\frac{S}{S_0}\right)$. [6 marks]

(ii) Using the function $I(S)$ show that an epidemic can occur only if the number of susceptibles S_0 in the population exceeds the threshold level b/a and that the disease stops spreading through lack of infectives rather than through lack of susceptibles. [6 marks]

(iii) For the trajectory which corresponds to $S_0 = \frac{b}{a} + \delta$, $I_0 = 0$, with δ small and positive, show that, to a good approximation, there are $(b/a - \delta)$ susceptibles who escape infection. [*The Kermack-McKendrick theorem*.] [8 marks]

[TOTAL 20 marks]

3. The Euler equations for rotation of a rigid body about its centre of mass, relative to axes fixed in the body, are

$$\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\omega_2\omega_3 &= G_1 \\ B\frac{d\omega_2}{dt} - (C - A)\omega_3\omega_1 &= G_2 \\ C\frac{d\omega_3}{dt} - (A - B)\omega_1\omega_2 &= G_3 \end{aligned}$$

where $\boldsymbol{\omega}(t)$, $\mathbf{G}$ are respectively angular velocity and applied torque and A, B, C are the principal moments of inertia.

(i) If $\mathbf{G} \equiv \mathbf{O}$ show that $L_1^2 + L_2^2 + L_3^2 = L^2$ (constant), where the angular momentum $\mathbf{L} = (A\omega_1, B\omega_2, C\omega_3)$, and that $\mathbf{L}$ must therefore lie on a sphere in (L_1, L_2, L_3) phase space. When $A > B > C$ show that there are six critical points on the phase sphere and show that four of these are centres and two are saddles. Explain the consequence of this result for assessing the stability of steady spin about a principal axis. [10 marks]

(ii) The 'SOLARIS' space probe has the shape of a surface of revolution so that its principal moments of inertia $A = B = 3C$. After encountering some Voyager space debris the probe is executing a tumbling motion with $\boldsymbol{\omega}(0) \neq \mathbf{0}$. Mr Spock attempts to stabilize the motion by applying a viscous torque

$$G = -(k\omega_1, k\omega_2, K\omega_3)$$

with k, K positive. Show that the angular velocity tends to alignment with the axis of dynamical symmetry as $\boldsymbol{\omega} \to \mathbf{0}$ if $\dfrac{k}{K}$ is greater than a critical value α, which is to be found. [10 marks]

[TOTAL 20 marks]

4. (i) Show that the one-dimensional maps

$$\begin{aligned} x_{n+1} &= r x_n (1 - x_n) \\ y_{n+1} &= s - y_n^2 \end{aligned}$$

where r, s are real parameters, may be related through a linear transformation $y_n = c x_n + d$, where $d(s)$ is given by $s = d + d^2$ and $c(s) = r(s) = -2d(s)$. [7 marks]

(ii) By working first with the x_n map, *or* otherwise show that:

(a) The y_n map has an asymptotically stable fixed point only when

$$s_1 < s < s_2,$$

where $s_1 = -1/4$ and $s_2 = 3/4$. [4 marks]

(b) The y_n map has an asymptotically stable two-cycle only when

$$s_2 < s < s_3,$$

where $s_3 = 5/4$. [4 marks]

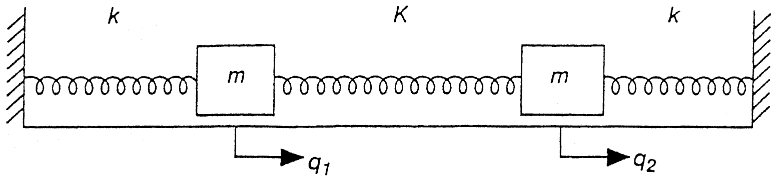
(iii) State briefly and qualitatively what would be expected to happen to the iterates of the y_n map for $s > s_3$. [5 marks]

[TOTAL 20 marks]

5. In the diagram below the two equal masses, m supported by a smooth table, are coupled by springs with differing spring constants, k, K so that the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{1}{2}\left[kq_1^2 + K\left(q_1 - q_2\right)^2 + kq_2^2\right] \\ &\equiv T - V, \end{aligned}$$

with q_1, q_2 the horizontal displacements of the masses from their equilibrium positions.



(i) Write down Lagrange's equations explicitly

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, 2)$$

for this mass/spring system and, in the case $K = 3k$, show that the normal frequencies of the horizontal motions of the masses are

$$\left(\frac{k}{m}\right)^{1/2} \qquad \text{and} \qquad \left(\frac{7k}{m}\right)^{1/2}.$$

[8 marks]

(ii) Also for the case $K = 3k$, find the general solution for $q_1(t)$, $q_2(t)$ in terms of the normal modes. By finding normal coordinates explicitly in terms of q_1, q_2, sketch the relative motions of the masses in each of the pure normal modes of oscillation. [8 marks]

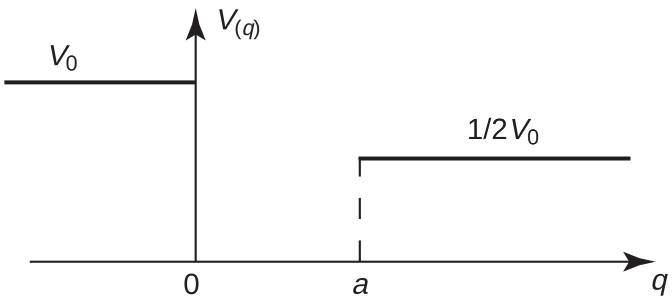
(iii) State briefly what would be expected to happen in the limit $\dfrac{K}{k} \to 0$ with k fixed? [4 marks]

[TOTAL 20 marks]

6. A particle of mass m moves along the q axis in the potential

$$V(q) = \left| \begin{array}{lll} V_0 & \text{for} & q < 0 \\ 0 & \text{for} & 0 \le q \le a \\ \frac{1}{2}V_0 & \text{for} & q > a \end{array} \right.$$

where V_0, a are positive constants.



(i) Sketch the phase plane portrait for this conservative system. [4 marks]

(ii) Find the action I for bounded oscillations in terms of the energy E. [6 marks]

(iii) Hence show that the period τ of the oscillations is then given by $\tau = a\left(\frac{2m}{E}\right)^{1/2}$. [4 marks]

(iv) If the potential well becomes narrower and deeper, on a time scale which is very long compared with τ and so that $q = V_0 a$ is exactly constant as a decreases very slowly, find how the energy and period of oscillation of the particle vary with a. Show that the particle must leave the potential well when a becomes sufficiently small.

[6 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

FOUNDATIONS OF QUANTUM MECHANICS

For Third-and Fourth-Year Physics Students

Tuesday 3rd June 2003: 10.00 to 12.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Give the definitions of:

(a) A *hermitian* operator acting in a Hilbert space $\mathcal{H}$. [1 mark]

(b) An *eigenvector* and *eigenvalue* of an operator. [1 mark]

Prove that:

(a) The eigenvalues of a hermitian operator are real numbers. [2 marks]

(b) A pair of eigenvectors corresponding to different eigenvalues are orthogonal. [2 marks]

(ii) A *projection operator* is defined to be any operator $\widehat{P}$ that is hermitian and that satisfies $\widehat{P}^2 = \widehat{P}$.

(a) Construct an explicit example of a projection operator in wave-mechanics (*i.e.*, a projection operator that acts on wave functions). [3 marks]

(b) Show that the eigenvalues of any projection operator can be only 0 or 1. [3 marks]

(iii) Let $\widehat{P}$ and $\widehat{Q}$ be a pair of projection operators.

(a) Show that if $\widehat{P}\widehat{Q}$ is a projection operator then $\widehat{P}$ and $\widehat{Q}$ must commute. [2 marks]

(b) Show that, conversely, if $[\,\widehat{P}, \widehat{Q}\,] = 0$ then $\widehat{P}\widehat{Q}$ is a projection operator. [2 marks]

(iv) Explain the role of projection operators in the spectral theorem for a hermitian operator. [4 marks]

[TOTAL 20 marks]

2. The spin degrees of freedom of a spin-1 particle are represented on the vector space $\mathbb{C}^3$ by the three matrices

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

An ensemble of such particles has the (unnormalised) state vector

$$|\psi\rangle = \begin{pmatrix} 1 \\ 3 \\ i \end{pmatrix}.$$

(i) (a) If the observable S_y is measured on this ensemble what is the average of the results obtained? [3 marks]

(b) What are the possible results of individual measurements of S_y? [3 marks]

(c) What are the probabilities of obtaining these results? Check that they sum to one. [10 marks]

(ii) The subensemble is selected of all those systems whose measurement of S_y gave the value 0. What is the average value of a measurement of S_x on this subensemble? [4 marks]

[TOTAL 20 marks]

3. (i) (a) Discuss carefully the extent to which the time evolution of a state vector in quantum theory can be said to be deterministic. [6 marks]

(b) What is the analogous situation in (non-chaotic) classical physics? [3 marks]

(ii) (a) A mixed state of a particular quantum-mechanical system at time t_1 is described by a density matrix $\widehat{\rho} := w_1 \widehat{P_1} + w_2 \widehat{P_2}$ where $\widehat{P_1}$ and $\widehat{P_2}$ are a pair of orthogonal projection operators, and the positive real numbers w_1 and w_2 satisfy $w_1 + w_2 = 1$. Show that it is impossible for $\widehat{\rho}$ to evolve according to the Schrodinger equation so that it becomes a projection operator at some later time t_2. [9 marks]

(b) What is the physical implication of this result? [2 marks]

[TOTAL 20 marks]

4. (i) Give the definition of the *uncertainty* $\Delta_{\vec{\psi}} A$ of a quantum mechanical observable A in a normalised state $\vec{\psi}$ and explain how this definition is motivated by ideas drawn from standard statistics.

[4 marks]

(ii) The Schwarz inequality implies that, for any vector $\vec{\psi}$ and pair of operators $\widehat{A}$ and $\widehat{B}$ we have

$$\left\| \widehat{A}\vec{\psi} \right\| \; \left\| \widehat{B}\vec{\psi} \right\| \; \geq \left| \left\langle \widehat{A}\vec{\psi}, \, \widehat{B}\vec{\psi} \right\rangle \right| .$$

Use this result to show that if O_1 and O_2 are any pair of physical observables then their uncertainties in a normalised state $\vec{\psi}$ satisfy the relation

$$\Delta_{\vec{\psi}} O_1 \Delta_{\vec{\psi}} O_2 \geq \frac{1}{2} \left| \left\langle \vec{\psi}, \left[\widehat{O}_1, \, \widehat{O}_2 \right] \vec{\psi} \right\rangle \right|$$

[7 marks]

(iii) A particle moves in a one-dimensional infinite potential well of width a such that the potential is 0 for $0 < x < a$. Compute the eigenvalues and eigenvectors of the energy operator $\widehat{H}$ and hence calculate the dispersion $\Delta_{\vec{\psi}} H$ of the energy in the state

$$\psi(x) := \sqrt{\frac{1}{a}} \left(\sin \frac{\pi x}{a} + \sin \frac{2\pi x}{a} \right).$$

[9 marks]

[TOTAL 20 marks]

5. (i) Motivate and explain clearly the definition of a function $f\left(\widehat{A}\right)$ where $\widehat{A}$ is any self-adjoint operator, and f is any real-valued function. (For simplicity you may assume that A is bounded and that its spectrum is discrete) [7 marks]

(ii) (a) Explain what is meant by a *value function* $V(A)$ of the physical quantities A in a quantum theory. [3 marks]

(b) If $\left[\widehat{A}, \widehat{B}\right] = 0$ show that $V(A + B) = V(A) + V(B)$, and $V(AB) = V(A)V(B)$. [4 marks]

(c) If $\widehat{P}$ is a projection operator show that $V(P) = 0$, or 1. What is the physical interpretation of this result? [2 marks]

(d) State the Kochen-Specker theorem and explain briefly its implications for the interpretation of quantum theory. [4 marks]

[TOTAL 20 marks]

6. Write short notes on THREE of the following

(i) The logical structure of classical physics.

(ii) The meaning of probability in quantum physics.

(iii) The reduction of the state vector.

(iv) Quantum entanglement.

[20 marks TOTAL]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

GENERAL RELATIVITY

For Third- and Fourth-Year Physics Students

Thursday 29th May 2003: 10.00 to 12.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

The Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

As usual, Latin indices $i, j, k \ldots$ run over 1, 2, 3 and Greek indices $\alpha, \beta \ldots$ run over 0, 1, 2, 3.

The line element is defined by $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

The Christoffel symbol is given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} (g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma}).$$

The Riemann tensor is

$$R^\beta{}_{\alpha\mu\nu} = \partial_\mu \Gamma^\beta_{\nu\alpha} - \partial_\nu \Gamma^\beta_{\mu\alpha} + \Gamma^\sigma_{\nu\alpha} \Gamma^\beta_{\mu\sigma} - \Gamma^\sigma_{\mu\alpha} \Gamma^\beta_{\nu\sigma}$$

The Ricci tensor is $R_{\alpha\nu} = \delta^\mu_\beta R^\beta{}_{\alpha\mu\nu} = R^\mu{}_{\alpha\mu\nu}$. In a free-fall frame

$$R_{\alpha\nu} = \frac{1}{2} g^{\rho\sigma} (g_{\sigma\nu,\,\alpha\rho} - g_{\alpha\nu,\,\sigma\rho} + g_{\alpha\rho,\,\sigma\nu} - g_{\sigma\rho,\,\alpha\nu})$$

1. (i) The Newtonian equation for the motion of a test particle in a gravitational potential φ is

$$\frac{d^2\mathbf{x}}{dt^2} + \nabla\varphi = 0, \tag{1.1}$$

where φ is related to the distribution of matter, mass density ρ_M, by

$$\nabla^2\varphi = 8\pi G\rho_M. \tag{1.2}$$

Why does Eq. (1.1) not depend on the mass of the particle? Explain the *principle* (without experimental detail) behind the experiments confirming this hypothesis. [4 marks]

(ii) In general relativity the equation of a geodesic for a particle is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0.$$

Show, in the weak-field, non-relativistic, static limit, that $\Gamma^i_{00} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$,
where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, whereby the spatial part of the geodesic equation reduces to

$$\frac{d^2x^i}{dt^2} + \frac{1}{2}c^2\frac{\partial h_{00}}{\partial x^i} = 0.$$

On comparison with Eq. (1.1) show that $h_{00} = 2\varphi/c^2$ in this limit. [6 marks]

(iii) The equation relating the metric to the energy-momentum tensor is

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$.
Show that this equation can be recast in the alternative form

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\sigma_{\ \sigma}).$$

[3 marks]

(iv) Given that $T_{\mu\nu} = (\rho + p/c^2)(U_\mu U_\nu/c^2) - (p/c^2)g_{\mu\nu}$, show that, in the same limit, the R_{00} equation correctly reproduces Eq. (1.2). Use the definitions given in the frontispiece.
[7 marks]

[TOTAL 20 marks]

2. (i) Under a general coordinate transformation $x^\mu \to x'^\mu$, a contravariant vector transforms in the same way as dx^μ. Similarly, a covector U_μ transforms in the same way as $\partial/\partial x^\mu$. Check that $U \cdot V$ is invariant. [3 marks]

(ii) (a) Show that, under a general change of coordinates $\partial_\mu V^\nu$ does not transform as a tensor. Why does it transform as a tensor under Lorentz transformations? [3 marks]

(b) The covariant derivative of a scalar is just the ordinary partial derivative, i.e. $D_\rho \varphi = \partial_\rho \varphi$. Explain the geometric significance of the additional term occurring in the expression

$$D_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\sigma\rho} V^\sigma$$

for the covariant derivative of a contravariant vector V^μ. [3 marks]

(c) From the scalar nature of $V \cdot U$ deduce that the covariant derivative of a covariant vector U_μ is

$$D_\rho U_\mu = \partial_\rho U_\mu - \Gamma^\sigma_{\rho\mu} U_\sigma .$$

You may assume that D_ρ satisfies the Leibniz rule $D_\rho(A_\alpha B^\beta) = (D_\rho A_\alpha)B^\beta + A_\alpha(D_\rho B^\beta)$ when acting on a product. [4 marks]

(iii) (a) The covariant derivative of $g_{\alpha\beta}$ is

$$D_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \Gamma^\sigma_{\alpha\mu} g_{\sigma\beta} - \Gamma^\sigma_{\beta\mu} g_{\sigma\alpha} .$$

By using the explicit form for $\Gamma^\sigma_{\rho\mu}$ given in the frontispiece, show that $D_\alpha g_{\mu\nu} = 0$. [4 marks]

(b) The action of $[D_\rho, D_\sigma]$ on a covariant tensor $T_{\beta\gamma}$ involves the Riemann curvature tensor in the form

$$[D_\rho, D_\sigma] T_{\beta\gamma} = -R^\alpha{}_{\beta\rho\sigma} T_{\alpha\gamma} - R^\alpha{}_{\gamma\rho\sigma} T_{\beta\alpha} .$$

By taking $T_{\beta\gamma} = g_{\beta\gamma}$ and using the result above, show that the totally covariant Riemann tensor $R_{\delta\beta\rho\sigma} = g_{\alpha\delta} R^\alpha{}_{\beta\rho\sigma}$ is antisymmetric in the first pair of indices. [3 marks]

[TOTAL 20 marks]

3. The hyperbolic plane is defined by the metric

$$ds^2 = y^{-2}(dx^2 + dy^2), \quad y \geq 0$$

(i) Using the fact that

$$L = y^{-2}(\dot{x}^2 + \dot{y}^2) = 1,$$

where $\dot{x} = dx/ds$, $\dot{y} = dy/ds$, write down the geodesic equations for $\dot{x}$, $\dot{y}$. [4 marks]

(ii) The equation of a semicircle ($y \geq 0$) of radius a with centre at x_0 on the x-axis is

$$(x - x_0)^2 + y^2 = a^2.$$

Write down an implicit equation for dy/dx. By comparing this with dy/dx obtained from (i) above show that the geodesics of the hyperbolic plane are arcs of semicircles with centres on the x-axis.

What happens as $a \to \infty$ with $a - x_0$ held constant? [6 marks]

(iii) Show that, for an arc of a semicircle of radius a, the interval s satisfies

$$\int ds = \int \frac{dy}{y\sqrt{1 - y^2/a^2}}$$

as y varies. [5 marks]

(iv) Deduce that the interval s for any geodesic beginning or ending on the x-axis is infinite. [2 marks]

(v) One of Euclid's postulates for conventional geometry in the (flat) plane is that, given a geodesic (straight line), and a point P, there is a unique geodesic (the parallel line) that passes through P and does not intersect the first geodesic. Show that this postulate is untrue for the hyperbolic plane. [3 marks]

[TOTAL 20 marks]

4. (i) Calculate the Christoffel symbols for the two-dimensional spaces with metrics given below. In each case a is a constant with the dimension of length.

(a)

$$ds^2 = \frac{a}{r} dr^2 + ar\, d\varphi^2,$$

(b)

$$ds^2 = dr^2 + a^2 e^{-2r/a}\, d\varphi^2.$$

[6 marks]

(ii) Explain why, in two dimensions, the Riemann tensor $R_{\gamma\alpha\mu\nu} = g_{\gamma\beta} R^{\beta}{}_{\alpha\mu\nu}$ has only one independent component.

[3 marks]

(iii) Calculate it for the metrics above. Is either space flat?

[11 marks]

Note: You may use the definition

$$R^{\beta}{}_{\alpha\mu\nu} = \partial_\mu \Gamma^{\beta}_{\nu\alpha} - \partial_\nu \Gamma^{\beta}_{\mu\alpha} + \Gamma^{\gamma}_{\nu\alpha}\Gamma^{\beta}_{\mu\gamma} - \Gamma^{\gamma}_{\mu\alpha}\Gamma^{\beta}_{\nu\gamma}$$

without derivation.

[TOTAL 20 marks]

5. The Schwarzschild metric governing the motion of planets around a star is

$$ds^2 = c^2\left(1 - \frac{a}{r}\right)dt^2 - \frac{dr^2}{1 - a/r} - r^2 d\varphi^2,$$

where we have restricted ourselves to the equatorial plane $\theta = \pi/2$.

(i) Write down the Euler-Lagrange equations for the geodesics in this space-time, using $L = (ds/d\tau)^2$.

(a) Show that the t equation gives $(1 - a/r)\dot{t} = \gamma$, where γ is a constant.

(b) Show that the φ equation gives $r^2\dot{\varphi} = J$, where J is a constant.

[4 marks]

(ii) Now show that the orbital equation for a planet going around a star, for which $L = c^2$, is of the form

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = au^3 + \alpha u + \beta,$$

where $u = r^{-1}$.

Identify the constants α and β in terms of γ and J.

[6 marks]

(iii) For circular orbits of radius $r = u^{-1}$, for which $du/d\varphi = d^2u/d\varphi^2 = 0$, α and β are given in terms of u as

$$\begin{aligned} \alpha &= 2u - 3au^2, \\ \beta &= -u^2 + 2au^3. \end{aligned}$$

Show that the corresponding values of γ and J are

$$\gamma = \frac{(1 - au)}{\sqrt{1 - 3au/2}}, \quad J = \sqrt{\frac{ac^2}{2u - 3au^2}}.$$

[4 marks]

(iv) If M is the mass of the star, then $ac^2 = 2GM$. Deduce that the coordinate time for one revolution of a circular orbit is the same as that for an orbit of radius r in Newtonian theory.

[3 marks]

(v) Show that the proper time experienced by the inhabitants of the planet in a complete orbit is $(1 - 3a/2r)^{1/2}$ times the corresponding coordinate time interval.

[3 marks]

[TOTAL 20 marks]

6. The line-element for a Schwarzschild black hole is

$$ds^2 = c^2\left(1-\frac{a}{r}\right)dt^2 - \frac{dr^2}{1-a/r} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

(i) What is the operational meaning of the coordinate r in this equation? Show that it is related to the radial distance, denoted ρ, by

$$dr = (1-a/r)^{1/2}d\rho.$$

[3 marks]

(ii) A clock is attached to a fixed spherical shell that encloses the black hole, with coordinate r corresponding to radial distance ρ. Its time is denoted by σ. Show that

$$\frac{d\rho}{d\sigma} = \frac{1}{(1-a/r)}\frac{dr}{dt}.$$

[3 marks]

(iii) Consider a space probe of mass m falling radially into the black hole, starting from rest with coordinate r_0, for which the radial distance is $\rho = \rho_0$. The geodesic equations in the presence of the hole are

$$\begin{aligned}(1-a/r)\dot{t} &= \gamma,\\ c^2\gamma^2 - \dot{r}^2 &= c^2(1-a/r),\end{aligned}$$

with $\dot{r} = 0$ when $r = r_0$. [Dots denote differentiation with respect to τ, where $ds^2 = c^2 d\tau^2$.] Deduce that $\gamma^2 = (1-a/r_0)$, whereby

$$\frac{d\rho}{d\sigma} = c\left(\frac{a/r - a/r_0}{1-a/r_0}\right)^{1/2}.$$

[6 marks]

(iv) The force needed to stop the probe at r_0 falling into the hole (i.e. to allow it to hover) is

$$F = -m\left.\frac{d^2\rho}{d\sigma^2}\right|_{r=r_0}.$$

Remembering that r_0 is constant, show that

$$F = \frac{mac^2}{2r_0^2}(1-a/r_0)^{-1/2}.$$

How does this differ from Newtonian gravity?

[8 marks]

Note: It may help to note that, for a function $f(r)$,

$$\frac{df(r)}{d\sigma} = f'(r)\frac{dr}{d\sigma} = f'(r)\frac{dr}{d\rho}\frac{d\rho}{d\sigma}.$$

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

GROUP THEORY

For Third-and Fourth-Year Physics Students

Monday 19th May 2003: 14.00 to 16.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

1. (i) Consider a set S with a binary composition law $*$ for its elements, and use them to define the concepts of (a) closure, (b) associativity, (c) commutivity, (d) *left* identity and (e) *left* inverse.

Give a definition of a group in terms of the left identity and left inverse.

Assuming that a left and right identity exist in a group, show they must be identical.

Assuming that a left and right inverse exist for all elements in a group, show they must be identical.

Show that in a group the identity is unique. [6 marks]

(ii) Without assuming the inverses need be unique, show that in a group the left cancellation law holds, i.e. $ab = ac$ implies $b = c$ and $ba = ca$. Deduce that the right cancellation law also holds.

Hence show that in a group the inverse element of any element of the group is unique.

Use the cancellation law to prove the rearrangement theorem, namely that left multiplication by one fixed group element is a one-to-one and onto map from the group G onto itself. [4 marks]

(iii) Consider a set of two elements $S = \{a, b\}$ and six possible binary composition laws as specified by the following multiplication tables:

$$(A)\begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & a \end{array}, \quad (B)\begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & a & a \end{array}, \quad (C)\begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & b & a \end{array},$$

$$(D)\begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & b \end{array}, \quad (E)\begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & a & b \end{array}, \quad (F)\begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & a & b \end{array}.$$

where for instance the table for (F) specifies that $a * b = b$.

Consider $b * b * a$ and use the results to identify which of the binary composition laws (A) - (F) are *not* associative.

Is this sufficient to prove which of the binary composition laws are associative?

Which of the binary composition laws (A) - (F) are *not* associative?

Which of the binary composition laws (A) - (F) are commutative?

For each of the binary composition laws (A) - (F) find if a, b or both are identities?

For each of the binary composition laws (A) - (F) find the left inverse of each element a, b of S or prove it does not exist. [8 marks]

(iv) You may assume, without further calculation, that in part (ii) of this question you have identified all the non-associative binary composition laws. Which of these multiplication laws makes S a group? Identify any isomorphisms between these groups. [2 marks]

[TOTAL 20 marks]

2. The quaternions can be defined as a set of eight elements,

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

with the following associative binary composition law:

(a) $1 * g = g * 1 = 1 \ \forall g \in Q$,

(b) The elements denoted as $-g$ are distinct from g and are given by $(-1) * g = g * (-1) = -g$ and $(-1) * (-g) = (-g) * (-1) = g$ where $g \in Q$,

(c) $i^2 = j^2 = k^2 = -1$ and $i * j = k$, $j * k = i$ and $k * i = j$.

Hint: in this question exploit the symmetry in the composition laws for i, j and k.

(i) Show that $j * i = -k$, $k * j = -i$ and $i * k = -j$. Hence show that Q with these composition laws is a group.
Define an abelian group. Is Q abelian? [4 marks]

(ii) Define what is meant when elements a and b of a group G are said to be conjugate.
Define a conjugacy class $[a]$ where $a \in G$.
Find the conjugacy classes of Q. [4 marks]

(iii) Define a proper subgroup. Find the proper subgroups of Q. Show they are each isomorphic to one of the cyclic groups Z_n. [4 marks]

(iv) Define a normal subgroup.
Show that normal subgroups are made of complete conjugacy classes.
Find the proper normal subgroups of Q

[4 marks]

(v) Define a product group.
Define a quotient group.
Find all possible quotient groups Q/H and show that they are isomorphic to a cyclic group or to a product group of cyclic groups.

[4 marks]

[TOTAL 20 marks]

3. (i) Define a homomorphism and an isomorphism.

Define a representation of a group in terms of a set of maps $\{\phi(g)\}$ from one set X to the same set X.

Define faithful and unfaithful representations in this language.

Define a linear representation in this context.

[4 marks]

(ii) Define a matrix representation. In terms of the general definition given above, what is the set X in this case? Are matrix representations linear or not and give an explanation.

Define a similarity transformation for a matrix. In terms of the space X, to what physical transformation does a similarity transformation correspond?

Show that one similarity transformation applied to all the matrices of matrix representation gives a further matrix representation of the same group.

What is a direct sum of matrices $\mathbf{D}^1 \oplus \mathbf{D}^2 \oplus \ldots$? Show that the direct sum of two matrix representations gives a third representation. Define reducible representations in the context of matrix representations of finite groups and in terms of similarity transformations.

[8 marks]

(iii) Prove that all matrix representations of finite group can be transformed into a unitary representation using a similarity transformation.

[8 marks]

[TOTAL 20 marks]

4. (i) Quote the first lemma of Schur.

Use it to prove part of the Great Orthogonality Theorem, namely that the matrices $\mathbf{D}(g)$ of dimension d of a unitary irreducible representation of a group G satisfy

$$\sum_{g \in G} [D(g)]_{ij} \, [D(g)]^*_{lm} = \frac{|G|}{d} \delta_{il} \delta_{jm}$$

Consider the following matrices:

$$\mathbf{D}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}(a) = i \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix},$$
$$\mathbf{D}(b) = -\mathbf{D}(e), \qquad \mathbf{D}(c) = -\mathbf{D}(a)$$

By constructing the vectors $\mathbf{V}_{ij} = (\mathbf{D}(e)_{ij}, \mathbf{D}(a)_{ij}, \mathbf{D}(b)_{ij}, \mathbf{D}(c)_{ij})$ for $i = 1$, $j = 2$ and $i = 2$, $j = 1$, show that this is not an irreducible representation. [10 marks]

(ii) Define a cyclic group. Prove that each element is in a class of its own, and prove that the irreducible representations are all one dimensional (you may quote without proof any theorems or lemmas you require).

Show that the matrices in (i) above form a representation of a cyclic group.

By forming the character table for the group, find which irreducible representations make up the two dimensional matrix representation given. [10 marks]

[TOTAL 20 marks]

5. In this question, if you use any relevant theorems, lemmas or properties you must indicate what they are but you need not prove them.

(i) The group of symmetry operations of the square is a representation of the group D_4. What are the symmetry operations of a square? What is the order of D_4?

Labelling the corners 1 to 4, explain why D_4 must be a subgroup of the permutation group S_4. Give an example of a permutation in S_4 which is not reproduced by a symmetry of the square.

Consider the different types of symmetry operation. Hence, or otherwise, show how many classes the group of symmetries of the square has.

How many irreducible representations does the square have and what must their dimensions be?

[6 marks]

(ii) Construct the character table for the group D_4.

[8 marks]

(iii) Four identical masses connected by identical springs, so that in the lowest energy configuration they form a stationary square. Consider the symmetry transformations of the eight real coordinates of the masses, $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$. By thinking about which vertices are left unchanged by the different symmetry transformations, find the character for this representation.

Is this representation reducible? If not deduce how many times each irreps appears in this representation.

[6 marks]

[TOTAL 20 marks]

6. Consider the set of all invertible operators $\{\widehat{S}^{\alpha}\}$ ($\alpha = 1, 2, \ldots, |G|$) which commute with a linear operator, $\widehat{M}$, so $[\widehat{S}, \widehat{M}] = 0$. Let $\{u_n(x)\}$, $n = 1, 2, \ldots, d$ be a maximal set of linearly independent eigenfunctions of the operator $\widehat{M}$, all with the same eigenvalue λ, so $\widehat{M} u_n(x) = \lambda u_n(x)$.

(i) Show that the set of operators $\{\widehat{S}^{\alpha}\}$ form a representation of a group G. [5 marks]

(ii) Why is $\sum_n \mathbf{R}_{mn} u_n(x)$ the most general form of an eigenfunction with this eigenvalue λ?
Show that each operator $\widehat{S}^{\alpha}$ maps one eigenvector to another of the same eigenvalue, $\widehat{S}^{\alpha} u_m(x) = \sum_n R^{\alpha}_{mn} u_n(x)$.
Hence show that R^{α}_{mn} form a representation of the same group. What is the dimension of this representation? [5 marks]

(iii) Suppose that the symmetry operation is a coordinate transformation $\mathbf{x} \to \mathbf{x}' = \mathbf{Q}\mathbf{x}$ so that

$$\widehat{S} u_n(\mathbf{x}, t) = u_n(\mathbf{x}', t) = u_n(\mathbf{Q}\mathbf{x}, t)\,.$$

What is the relationship between eigenfunctions which transform under one-dimensional representations and the characters? [2 marks]

(iv) Consider a particle moving in two dimensions (x, y) in a potential V where

$$\widehat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x, y), \quad V(x, y) = V(-x, y) = V(x, -y).$$

Show that the symmetry group is $Z_2 \times Z_2$. Write down the character table for this group. Hence describe the degeneracy of the energy eigenstates forced by symmetry. What does symmetry tell us about the form of eigenfunctions lying in each of the different one-dimensional irreducible representations? [8 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

LASERS, OPTICS & HOLOGRAPHY

For Third- and Fourth-Year Physics Students

Tuesday 20th May 2003: 14.00 to 16.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Determine the form of the intensity pattern at a large distance r due to diffraction of a plane wave by the following apertures:

(a) A single narrow slit aperture of width $2a$ (you may assume that this is infinitely long).

(b) A rectangular aperture with sides of lengths $2a$ and $2b$. (You may assume that $r \gg a, b$.)

(c) For a rectangle of sides $150 \times 600\,\mu m$ illuminated by plane light at a wavelength of 550 nm at what angle from the optic axis are the first minima found in the two orthogonal directions parallel to the edges of the aperture.

[10 marks]

(ii) A rectangular aperture of sides $100 \times 800\,\mu m$ is the "object" in a 4-f optical processing system with large aperture lenses. The processing system uses plane wave coherent light of wavelength 632.8 nm, the lenses of the 4-f system all have focal length of 1 m, and there is a small circular aperture of diameter 4 mm in the Fourier plane that acts as a filter.

(a) What is the highest spatial frequency in the image?

(b) Sketch the effect upon the image of this filter.

(c) What is the effect upon the image of increasing or decreasing the illuminating wavelength?

(d) What filter aperture should be used to ensure that the edges of the image are almost as sharp as those in the original object.

[10 marks]

[TOTAL 20 marks]

2. (i) Write down the Ray Transfer Matrix that represents the action upon rays propagating close to the optic axis of:

(a) Free space propagation over a distance d.

(b) A thin lens of focal length f.

(c) From these results determine the Ray Transfer Matrix for the combination of free-space propagation over a distance d followed by a thin lens of focal length f. [5 marks]

(ii) An optical cavity comprises a pair of mirrors (of radii of curvature R_1 and R_2) which are separated by a distance d. We define the parameters g_1 and g_2 as:

$$g_1 = 1 - d/R_1 \quad and \quad g_2 = 1 - d/R_2.$$

It can be shown that for stable solutions only certain values of the product of these parameter $g_1 g_2$ are permitted.

(a) State the constraints on the value of $g_1 g_2$.

(b) Plot in a graph of g_1 versus g_2 the regions where the cavity will be stable.

(c) Which of the following cavities are stable, marginally stable and unstable:

I. $R_1 = -4\,\text{m}$, $R_2 = \infty$, $d = 1\,\text{m}$

II. $R_1 = R_2 = \infty$, $d = 40\,\text{cm}$

III. $R_1 = 0.5\,\text{m}$, $R_2 = 1\,\text{m}$, $d = 0.5\,\text{m}$

IV. $R_1 = 50\,\text{cm}$, $R_2 = 75\,\text{cm}$, $d = 1\,\text{m}$

Plot their positions on the stability diagram. [10 marks]

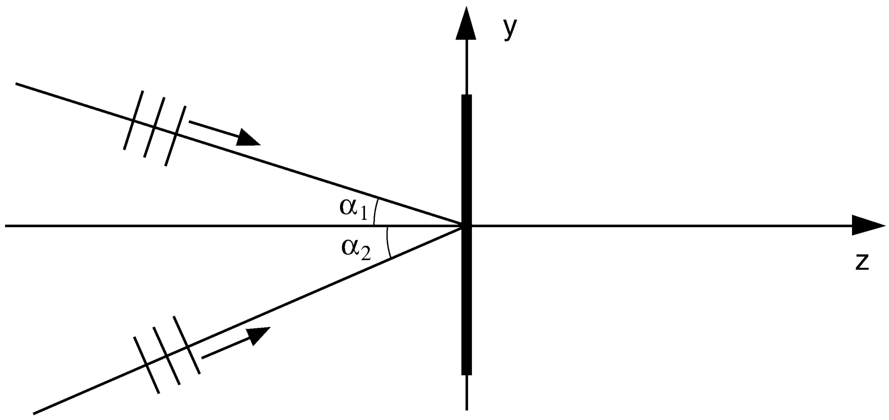
(iii) Consider a laser operating at 1064 nm with a stable cavity of length 1.2 m for which the lowest order laser mode has a minimum waist of 1.3 mm diameter at the centre point of the cavity. There is an output coupling mirror of reflectivity 0.95 (assume the other mirror has reflectivity 1.0).

(a) What is the diameter of the beam waist 2m beyond the output coupler?

(b) Estimate by what factor the intensity 2 m beyond the output coupler will be reduced relative to the intensity at the minimum waist in the cavity? [5 marks]

[TOTAL 20 marks]

3. (i) Two coherent light beams with the same intensity and of wavelength λ are incident upon a recording screen at $(y, 0)$ as shown in the figure.



(a) Show that the intensity at the screen is given by:

$$I(y,0) = I_o \cos^2\left(\frac{\pi y}{\lambda}(\sin\alpha_1 + \sin\alpha_2)\right)$$

where I_o is a constant.

(b) If $\alpha_1 = \alpha_2 = \alpha$, what value of 2α gives fringes in the screen separated by $1\,\mu$m for illumination with 630 nm wavelength light?

[10 marks]

(ii) (a) Briefly describe the steps required to record, develop and reconstruct a hologram.

(b) Consider the "hologram" formed for the case outlined above in (i)(a) (think of beam 1 as the reference and beam 2 as the object). Determine the diffraction from the hologram. By considering the properties of the different order of diffraction ($m = 0, +1$and -1), identify the diffracted beam that corresponds to the original "object" beam.

[8 marks]

(iii) Outline how holography can be used to measure small motions of a moving surface.

[2 marks]

[TOTAL 20 marks]

4. (i) (a) The polarisation P of a dielectric medium can be expressed in terms of a power series expansion in the electric field strength E of the applied field:

$$P = \varepsilon_0 \left(\chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \ldots \right) . \quad (4.1)$$

This is referred to as the polarisation expansion.
Under what conditions will the first order term dominate?

(b) By considering the medium as an ensemble with a number density N of bound electronic charges, each acting as an oscillator of natural frequency ω_0, the linear response of the medium to an electromagnetic field ($E(t)$ at frequency ω) can be determined in terms of the classical displacement of the charges by the field. This treatment yields

$$P = eNx(t) = \frac{e^2/m_e}{\left(\omega_0^2 - \omega^2\right)} N E(t) . \quad (4.2)$$

Given that the linear susceptibility of the medium is related to the refractive index n by the expression $n^2 = 1 + \chi^{(1)}$ determine an expression that gives the frequency dependence of n^2.

(c) Generalise equation 4.2 to the case of a real medium of atoms which have many resonances each with an oscillator strength f_j and a radiative damping γ_j. [7 marks]

(ii) (a) Sketch the refractive index dependence upon the electromagnetic field frequency for the case of a typical atomic gas indicating where the dipersion is "normal" (i.e. increases with frequency) and where the dispersion is "anomalous" (i.e. decreases with frequency).

(b) By employing a modified version of the model used above (part (i)(b)) briefly explain the phenomenon of birefringence. [5 marks]

(iii) For the high intensities available in a laser field the second order term in the polarisation expansion (equation 4.1) may become important.

(a) What is the condition on the medium symmetry for there to be a non-vanishing second order contribution?

(b) Show that in this circumstance if a field at frequency ω is applied that the second harmonic will be generated.

(c) What constraint is placed upon the refractive index of the medium by the need to conserve momentum in this process?

(d) How can this constraint upon the refractive index be satisfied in a birefringent crystal? [8 marks]

[TOTAL 20 marks]

5. (i) (a) Briefly describe the energy level scheme pertinent to laser action in the Nd:YAG laser (use a sketch if required).

(b) What type of light sources are used for pumping inversion in this system for pulsed or CW operation?

(c) Describe the temporal behaviour of light emission from a simple Nd:YAG laser cavity comprising just mirrors and the gain medium in the case where the pump comprises a pulsed light source capable of pumping inversion in the Nd:YAG crystal by a factor of many times that required for reaching the threshold in the cavity that contains no additional intra-cavity elements.

[8 marks]

(ii) (a) Briefly explain how the output pulse from a Nd:YAG laser can be controlled so as to maximise the output pulse peak power. Identify any intra-cavity optical elements are required to achieve this.

(b) For a cavity of length 250 mm and effective mirror reflectivity $R = 0.92$ containing a Nd:YAG crystal in which a total energy of 0.75 J is stored, what will be the duration and peak power of the output pulse?

[8 marks]

(iii) The second-harmonic of a Q-switched Nd:YAG laser is used to pump a ring laser cavity containing a titanium sapphire crystal and no other intra-cavity elements. This titanium sapphire laser is to be used for the generation of pulses of single mode light.

(a) Explain how single mode operation can be attained in such a cavity if a low power narrow-bandwidth CW laser is also available.

(b) The pulse length of this cavity is typically >5 times longer than the pump laser used, estimate the minimum bandwidth of the output of the titanium sapphire ring laser.

[4 marks]

[TOTAL 20 marks]

6. Write short notes on THREE of the following. Use sketches and state important mathematical expressions where necessary.

(i) The resolution limits to human vision.

(ii) The longitudinal modes of a Fabry-Perot etalon.

(iii) Active and passive laser mode-locking techniques.

(iv) The physical origin of non-linear refractive index.

(v) External cavity stabilisation of a laser diode.

(vi) Phase contrast microscopy.

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

MOLECULAR BIOPHYSICS

For Third- and Fourth-Year Physics Students

Wednesday 28th May 2003: 14.00 to 16.00

Answer THREE questions, taking at least ONE question from Section A and at least ONE question from Section B.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Given below is the sequence of a short peptide as well as a number of mutant forms of the protein:

```
normal peptide     Met-Trp-Tyr-Arg-Gly-Ser-Pro-Thr
mutant 1           Met-Trp
mutant 2           Met-Cys-Ile-Val-Val-Leu-Gln
mutant 3           Met-Trp-Tyr-Arg-Gly-Ser-Pro-Met
```

Describe, in each case, the general type of mutation and indicate the minimum base changes.

[8 marks]

(ii) The following sequence belongs to a segment of DNA that is being replicated and the arrow indicates the direction of progression of the replication fork.

```
(1) 5′ GAATGCTAATGAAAGCAACTAAACTAGTACTTGGGGCGGTGATTTAACCGCA
(2) 3′ CTTACGATTACTTTCGTTGATTTGATCATGAACCCCGCCACTAAATTGGCGT
                                   ⟶
```

Which strand would act as the template for leading strand synthesis?

Which one as a template for lagging strand synthesis?

If the same segment of DNA was being transcribed and the arrow indicated the direction of transcription:

- which strand would act as a template?
- what would be the sequence of the corresponding mRNA?

In the above mRNA identify:

- the start point for translation
- the end point for translation
- write down the amino-acid sequence encoded by the mRNA

[12 marks]

[TOTAL 20 marks]

Genetic Code - mRNA to protein

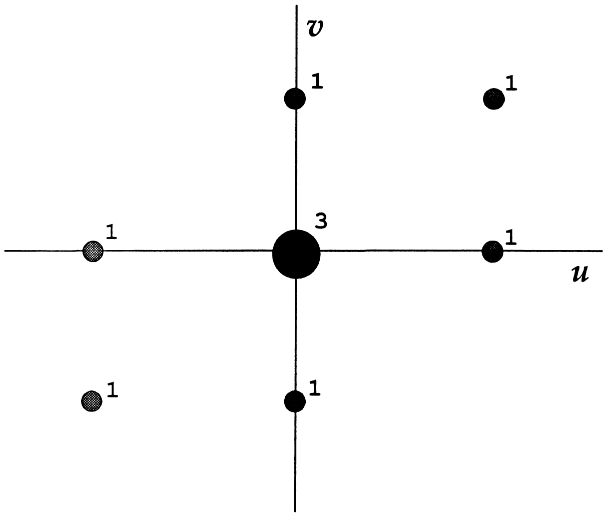
```
              Second position
        |  U  |  C  |  A  |  G  |
    ----+-----+-----+-----+-----+----
        | Phe | Ser | Tyr | Cys | U
      U | Phe | Ser | Tyr | Cys | C
F       | Leu | Ser | Stop| Stop| A    T
i       | Leu | Ser | Stop| Trp | G    h
r    ---+-----+-----+-----+-----+--   i
s       | Leu | Pro | His | Arg | U    r
t     C | Leu | Pro | His | Arg | C    d
        | Leu | Pro | Gln | Arg | A
P       | Leu | Pro | Gln | Arg | G    P
o    ---+-----+-----+-----+-----+--   o
s       | Ile | Thr | Asn | Ser | U    s
i     A | Ile | Thr | Asn | Ser | C    i
t       | Ile | Thr | Lys | Arg | A    t
i       | Met | Thr | Lys | Arg | G    i
o    ---+-----+-----+-----+-----+--   o
n       | Val | Ala | Asp | Gly | U    n
      G | Val | Ala | Asp | Gly | C
        | Val | Ala | Glu | Gly | A
        | Val | Ala | Glu | Gly | G
    ----+-----+-----+-----+-----+----
```

2. Answer THREE out of the following four questions:

(i) You want to test by hybridisation whether a DNA double strand containing 10,000 base-pairs includes a particular sequence. Calculate the minimal length of the nucleotide to use in the hybridisation to ensure that there is less than 1% chance of a match occurring by chance.

(ii) Define the Patterson function in terms of autoconvolution of the electron density and explain why the resulting map represents all the interatomic vectors.

Draw a molecule that would produce the Patterson map shown in the figure below, assuming that all atoms have the same atomic number. Is this molecule unique?



(iii) Describe how X-rays are generated and how they interact with matter.

(iv) Write notes on the process of transcription, and give an example of how gene expression can be regulated by the cellular environment.

[TOTAL 20 marks]

3. (i) Discuss the phase problem in crystallography, mentioning the methods used to solve the problem for small molecules and for macromolecules. Give a brief summary of the MIR method.

[10 marks]

(ii) A protein:heavy atom complex contains three mercury atoms ($Z_{\mathrm{Hg}} = 80$) at positions: (0.1, 0.1, 0.0), (0.2, 0.0, 0.1), (0.1, 0.3, 0.1), in fractional coordinates.

Calculate the heavy atom contribution for the reflection (1, 1, 2) and illustrate your answer with a vector diagram.

Assume that the atomic scattering factor can be approximated by the atomic number . Why is this a valid approximation?

If the intensities of the (1, 1, 2) reflection are $I_P = 78,400$ for the native crystal and $I_{PH} = 167,608$ for the protein:heavy atom complex, determine the phase angle of the reflection for the protein crystal.

[10 marks]

[TOTAL 20 marks]

SECTION B

4. (i) According to transition state theory the isomerisation reaction

$$X \xrightarrow{k} Y$$

may be represented as

$$X \xleftrightarrow{K^{\neq}} X^{\neq} \xrightarrow{\nu} Y$$

where $X^{\neq}$ is the transition state, $K^{\neq}$ is the equilibrium constant for the interconversion of X and $X^{\neq}$ and ν is the rate constant for conversion of $X^{\neq}$ to product, Y. Assuming that $h\nu = k_B T$, show that the rate constant for the reaction may be expressed as

$$k = \frac{k_B T}{h} e^{-\Delta G^{\neq}/RT}$$

where R is the universal gas constant. What is the meaning of the exponential term in the expression?

[8 marks]

(ii) Discuss catalytic antibodies in relation to the above analysis.

[6 marks]

(iii) For an enzyme-catalysed reaction the catalytic rate constant is expressed as

$$k_{cat} = \frac{k_B T}{h} e^{-\Delta G^{\neq}_{cat}/RT}$$

A bacterial enzyme with $k_{cat} = 1.5\,\mathrm{s}^{-1}$ at $25^{\circ}\,\mathrm{C}$ is mutated for commercial use so that the standard enthalpy of activation is reduced from 10 to $2.5\,\mathrm{kJ\,mol^{-1}}$ without affecting the standard entropy of activation. What is the rate constant for the mutated enzyme if it is used at $40^{\circ}\,\mathrm{C}$?

($R = 8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$; $k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$; $h = 6.63 \times 10^{-34}\,\mathrm{J\,s}$)

[6 marks]

[TOTAL 20 marks]

5. (i) What are the essential features of enzymes? Illustrate your answer with reference to the protease, chymotrypsin.

[6 marks]

(ii) (a) Consider the following reaction scheme in which an enzyme E catalyses the reaction of two substrates, X and Y to form the product P.

$$\begin{array}{rcl} E \;+\; X & \overset{K_X}{\rightleftharpoons} & EX \\ + & & + \\ Y & & Y \\ K_Y \Big\updownarrow & & \Big\updownarrow K'_Y \\ EY \;+\; X & \underset{K'_X}{\rightleftharpoons} & EXY \xrightarrow[k_{cat}]{} P \end{array}$$

K_X, K_Y, K'_X and K'_Y are the dissociation constants for the binding steps shown in the reaction scheme and k_{cat} is the turnover number for the reaction.

Show that $K_Y K'_X = K_X K'_Y$.

[4 marks]

Show further that the rate of formation of product, v, is given by:

$$v = \frac{v_{max}}{1 + \frac{K'_X}{[X]}\left(1 + \frac{K_Y}{[Y]}\right) + \frac{K'_Y}{[Y]}}$$

where v_{max} is the product of k_{cat} and the total enzyme concentration.

[6 marks]

(iii) On a double-reciprocal Lineweaver-Burke plot sketch the difference between the reactions carried out over a range of concentrations of X at low and high concentrations of Y. You may assume here that $K_Y = K'_Y$.

[4 marks]

[TOTAL 20 marks]

6. (i) Describe briefly the properties of myoglobin that account for its function as an oxygen storage molecule.

[5 marks]

(ii) Derive an expression for Y the fractional occupancy of binding sites on myoglobin in terms of the oxygen partial pressure (pO_2) and the partial pressure at which the molecule is 50% occupied (p_{50}).

[5 marks]

(iii) Using this expression, show that the net fractional transfer (ΔY) between two locations of oxygen partial pressures p_1 and p_2 may be expressed as:

$$\Delta Y = \frac{p_{50}(p_1 - p_2)}{(p_1 + p_{50})(p_2 + p_{50})}$$

Sketch on a graph the variation in ΔY as a function of p_{50} assuming that the molecule is operating between two regions with oxygen partial pressures of 0.02 and 0.1 atm.

[5 marks]

(iv) Discuss the significance of your findings with regard to different roles of myoglobin and haemoglobin in oxygen transport.

[5 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

OPTICAL COMMUNICATIONS PHYSICS

For Third - and Fourth - Year Physics Students

Tuesday 3rd June 2003: 14.00 to 16.00

Answer ALL parts of Section A and TWO questions from Section B.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. State Shannon's formula for channel capacity, clearly defining the terms used and the assumptions made for its validity.

[2 marks]

A 3 minute analogue music clip having frequencies from 200 Hz to 15000 Hz and with a signal to noise power ratio of 40 dB undergoes analogue to digital conversion.

(i) What is the *minimum* number of bits of information in the digital signal required to encode the information of the analogue signal?

[4 marks]

(ii) How long would it take to transmit this information as a binary digital code in a medium with a bandwidth of 10 kHz? How can real-time digital transmission be achieved with this transmission medium?

[4 marks]

(iii) Explain what is meant by additive white Gaussian noise (AWGN).

[2 marks]

(iv) With the aid of suitable diagram(s) describe how the bit error rate of a digital signal due to AWGN can be quantified. [Detailed mathematical calculation is not required].

[4 marks]

(v) Describe the two main types of copper cabling for communication links and explain why the attenuation of the copper increases at high frequencies.

[4 marks]

[TOTAL 20 marks]

SECTION B

2. (i) Explain the problems and solutions to how a mobile telecommunications network handles:

(a) Limited network bandwidth. [4 marks]

(b) Signal degradation associated with free-space signalling. [3 marks]

(ii) Describe, in principle, how a baseband digital signal with frequencies from 0 – 25 kHz can be formatted for free-space communications at central frequency $\nu_0 = 900\,\mathrm{MHz}$. In your answer you should provide a mathematical description of the method in the time and frequency domains and provide sketches of the baseband and mobile signals in the frequency domain with clear scales on the frequency axis. [7 marks]

(iii) A base station in the mobile network emits a signal power of 1 kW uniformly in all directions. A piece of mobile equipment has an effective collection area of $5\,\mathrm{cm}^2$ and requires a power of $1\,\mu\mathrm{W}$ for good reception. What is the maximum range for good reception? [6 marks]

[TOTAL 20 marks]

3. (i) An optical fibre link utilises a narrow bandwidth modulated laser source with an average output power of 1 mW emitting at 1.55 μm. The fibre loss is known to be 0.4 dB/km and the receiver requires a minimum power of -25 dBm in order to achieve an acceptable bit error rate.

(a) Calculate the attenuation-limited length of the link. [2 marks]

(b) Erbium Doped Fibre Amplifiers (EDFAs) might be employed to increase this value. Briefly describe the operation of an EDFA and how this might be incorporated into a system. [4 marks]

(c) Indicate the limitations of such amplifiers for Wavelength Division Multiplexing (WDM) applications. [2 marks]

(d) Briefly outline how dispersion in silica fibres can also limit the length of the link. [3 marks]

(ii) The same fibre is to be assessed for operation at 1.3 μm using the cut- back method to determine the loss at this wavelength. This involves a short length, 2 km, of fibre and an optical receiver. For a particular laser source operating at 1.3 μm the measured output voltage at the receiver was found to be 2.7 V. The fibre was then cut to a length of 2 m and the output voltage increased to 11.2 V. The same source was used in both cases and the receiver is known to be linear in incident optical power. Using this information, calculate the attenuation of the fibre in dB/km at this wavelength. [4 marks]

(iii) Account for the difference between the values of attenuation measured at 1.55 μm and 1.3 μm by outlining the main factors which give rise to signal loss in silica fibres. [5 marks]

[TOTAL 20 marks]

4. (i) Thermal and shot noise in photodiode detectors is frequently described as Gaussian White noise. Explain why this is so by considering the origin of each type of noise. For each case derive expressions for the mean square current.

[6 marks]

(ii) Show that the photocurrent, I_P, generated by a photodiode receiver may be written

$$I_P = \frac{\eta P_0 q \lambda}{hc}$$

where η is the quantum efficiency, q the electronic charge, λ the wavelength of the detected photons and P_0 the received optical power.

[4 marks]

(iii) An InGaAs p-i-n photodiode with a quantum efficiency of 0.7 and a negligible dark current is used to detect $1.3\,\mu$m light. Calculate the output current, I_P, if the average optical power incident on the detector is $-20\,$dBm.

[4 marks]

(iv) Using your expressions derived in part (i) show that the Signal-to- Noise ratio (SNR) for the receiver is given by

$$\mathrm{SNR} = \frac{{I_P}^2}{2q\,(I_P + I_D)\,\Delta f + 4k_B T \Delta f / R_L}$$

where T is the ambient temperature, R_L, the load resistor, I_D the dark current and Δf the receiver bandwidth.

[2 marks]

(v) Calculate the shot noise limited SNR (in dB) of the detector described in part (iii) if this detector is used in a 100 Mb/s digital communications link which uses non-return-to-zero coding.

[4 marks]

[TOTAL 20 marks]

5. Briefly discuss THREE of the following: [All parts carry equal marks]

(i) Data compression techniques.

(ii) Dispersion in optical fibres.

(iii) Error detection and correction.

(iv) Optical sources for communications.

(v) Electro-optic modulators for optical switching.

(vi) Packet-switching in networks and its advantages/disadvantages.

[20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

PHYSICS APPLIED TO MEDICINE

For Second -Third - and Fourth -Year Physics Students

Thursday 22nd May 2003: 14.00 to 16.00

Answer All parts of Section A and TWO questions from Section B.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) All diagnostic X-ray tubes use a metallic filter in order to limit the spectral range contained in the X-ray beam. Estimate the thickness of Al required to reduce all X-ray components with an energy $E < 10\,\text{keV}$ by a factor of 100.

[$\mu_{\text{Al}}(E = 100\text{keV}) = 0.4\,\text{cm}^{-1}$.
You should assume that photoelectric absorption is dominant.] [3 marks]

(ii) Explain what is meant by the terms *functional* and *anatomical* imaging . Describe briefly the diagnostic role of Gamma imaging and X-ray CT in terms of these two broad categories. [5 marks]

(iii) Ultrasound investigations of the mammalian brain are technically very much more difficult than investigations of the foetus. Explain this using simple ideas of impedance matching. [5 marks]

(iv) Estimate the total energy, in joules, deposited in a human subject after a 500 MBq Bolus of ^{15}O has been injected for a PET investigation. Explain briefly why this energy is not uniformly distributed throughout the body.
[^{15}O *emits a positron with energy* 1.68 MeV *with a half life of* 120 s.] [3 marks]

(v) MRI uses different RF pulse durations in the course of an imaging procedure. Estimate the pulse duration required to perform a uniform 90° spin rotation on a 10 cm thick slice of patient, using a field gradient strength of 3 mT/m. [4 marks]

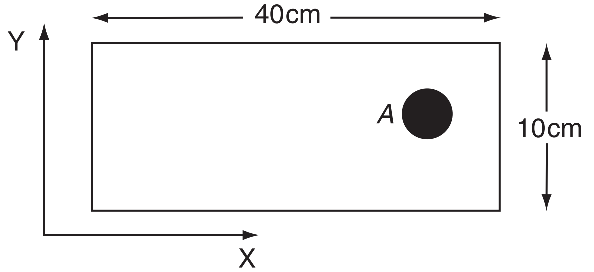
[TOTAL 20 marks]

$\left[1\text{eV} = 1.60 \times 10^{-19}\text{J}\right]$

SECTION B

2. (i) Define the term *projection* in the context of X-ray CT. [2 marks]

The figure shows a simplified model of a cross section of a human body.



A is an object with diameter 0.5 cm and an absorption coefficient, $\mu = 0.8\,\text{cm}^{-1}$. The bulk of the cross section is assumed to be uniform with $\mu = 0.25\,\text{cm}^{-1}$.

(ii) Determine the change in primary image intensity caused by object A in single projections taken parallel to (a) x axis and (b) y axis [2 marks]

(iii) Comment on the visibility of object A in these projections. In which direction would you expect the effects of scatter to be the more important? [2 marks]

A line by line, **rectilinear** CT scan of this cross section is planned with the main objective being to image object A.

(iv) Explain briefly why A should become more visible in a reconstructed image. [3 marks]

(v) What is the maximum size of reconstruction voxel which should be used? [2 marks]

(vi) Estimate the number of projections required to reconstruct the image. [3 marks]

(vii) Draw a labelled sketch of the K space map after 4 projections have been obtained. You should show clearly the sampling intervals involved. [3 marks]

(viii) Explain briefly how the modern, fan beam CT geometry greatly speeds up the data acquisition and improves spatial resolution. Why is a reduced data acquisition time desirable in any imaging procedure?

[3 marks]

[TOTAL 20 marks]

3. (i) Write down the fundamental equation of MRI and explain how with an appropriate sequence of operations a sampled map of K space, corresponding to this equation, can be obtained.

[5 marks]

A typical 1T scanner uses an X gradient of 5 mT/m when imaging human cross sections with a FOV measuring 220×220 mm digitised into 512×512 voxels.

(ii) Determine the maximum frequency along the x direction after detection.

(iii) Estimate a suitable measuring time for the acquisition of each K space line, given a required spatial resolution of 1 mm.

(iv) What is the minimum sampling frequency required? Explain briefly why this is important.

(v) Estimate the total time required to collect sufficient data for this slice in a conventional imaging scheme, given that T1 contrast is to be suppressed.

[8 marks]

In Fast MRI schemes considerable effort is made to acquire data close to the origin of K space, $K_x = 0$, $K_y = 0$ early in a scan.

(vi) Explain why the origin of K space is particularly important in all tomographic imaging and why this is particularly relevant to MRI.

[4 marks]

(vii) Draw a labelled diagram showing how an appropriate sequence of MRI, K space lines could be acquired using phase encoding gradient pulses. Explain briefly why in fast imaging schemes T1 contrast can never be totally suppressed.

[3 marks]

[TOTAL 20 marks]

4. (i) Outline the mechanisms that contribute to the attenuation of X-rays in the energy range of 10 – 200 keV in human tissues and indicate clearly which are the most important over the energy range used for medical X-ray imaging applications. How does the relative contribution of the different mechanisms change with tissue type and with energy? [6 marks]

(ii) Ultrasound in the frequency range 1 – 15 MHz is used for medical diagnostic applications. Describe the processes which contribute to the attenuation of the ultrasound beam. Indicate which are most important and also indicate what characteristics of the body tissues determine their response to the ultrasound wave. [7 marks]

(iii) In some investigations using X-ray imaging or MRI a contrast agent is used. Explain why the use of such an agent can alter the image in each case. Why is the use of such agents more common in X-ray than in MRI?

In an X-ray image of part of the circulation system the aim is to detect blood vessels of diameter 1 mm. Using an attenuation coefficient for blood of $0.17\,\mathrm{cm}^{-1}$, and $0.18\,\mathrm{cm}^{-1}$ for soft tissue estimate for the proportion of iodine that should be added to the blood to meet this requirement. State clearly the assumptions made in obtaining this estimate.
$[Z_{\mathrm{Iodine}} = 53]$ [7 marks]

[TOTAL 20 marks]

5. (i) Medical X-ray images are often made using an appropriate photographic film as the detector. Outline how the actual X-ray image departs from the ideal in such cases and what are the sources of noise in the final image. What strategies are adopted to reduce the noise in the final image, and what are their limitations?

[7 marks]

(ii) In diagnostic ultrasound imaging a single instrument is commonly used as source and detector for the ultrasound. Outline the three types of scan that can be obtained using such a system.

[3 marks]

(iii) In an image model for ultrasound interacting with an ideal target the reflected waveform, $g(x, y)$, can be expressed by,

$$g(x, y) = h_1(y) \otimes h_2(x) \otimes f(x, y).$$

Indicate what each term in this expression represents.

This model omits several features of the ultrasound beam characteristics and its interactions. Indicate clearly both the idealising assumptions made and the departures from these assumptions in the formation of an ultrasound image.

[7 marks]

(iv) For a transmitter emitting sound at a frequency of f_0 Hz, moving at velocity v with respect to an observer, the frequency difference between the source and the detected frequency is given by

$$f_{return} - f_0 = \Delta f \simeq \pm \frac{fv}{c},$$

where c is the velocity of sound in air. Explain how this expression is modified to describe the frequency shift in Doppler ultrasound imaging.

[3 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

PLASMA PHYSICS

For Third- and Fourth-Year Physics Students

Thursday 22nd May 2003: 14.00 to 16.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

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Hand in THREE answer books even if they have not all been used.

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Turn over for questions

1. (i) A test particle of charge, Q is placed in a plasma of electron density, n_e, which has an electron temperature, T_e and an ion temperature T_i. Show that the electromagnetic potential, ϕ, set up in the plasma at a distance r is approximately given by the expression (for $k_B T_e, k_B T_i \gg e\phi$);

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{\phi}{\lambda_D^2}$$

Obtain an expression for λ_D and discuss its physical significance. [6 marks]

(ii) Use the substitution $u = r\phi$ to solve this equation and sketch this potential. [5 marks]

(iii) What is the total charge enclosed in a radius λ_D ? Sketch the ion and electron charge density (for $k_B T_e, k_B T_i \gg e\phi$). [9 marks]

[TOTAL 20 marks]

Note: in spherical co-ordinates (r, θ, ϕ)

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

2. (i) What is an adiabatic invariant in a dynamical system? Define the "first" and "second" adiabatic invariants in plasma physics? Under what conditions are they approximately constant?

[6 marks]

(ii) What is a magnetic mirror? How can it be used as a magnetic confinement device? [4 marks]

(iii) Using the first adiabatic invariant, μ show that the velocity, v_z of an electron in a magnetic field, B_z which varies in the z direction is given by;

$$v_z = \left(v_0^2 - \frac{2\mu B_z(z)}{m_e} \right)^{1/2}$$

where v_0 is a constant.

[3 marks]

(iv) If the magnetic field in a magnetic mirror is given by

$$B_z(z) = B_0 \frac{|\, z \,|}{l}$$

such that l is a constant, calculate the second adiabatic invariant. If l increases at a rate R, determine a condition on R such that the second adiabatic invariant is conserved.

[7 marks]

[TOTAL 20 marks]

3. (i) The Rutherford scattering formula for two charged particles is given by:

$$\cot\left(\frac{\chi}{2}\right) = \frac{4\pi\varepsilon_0 m v^2 b}{q_1 q_2}$$

Indicate χ, b, and v on a sketch.

[4 marks]

(ii) Show that the collision time (i.e., the time required to produce a root mean square deflection of 90°) for multiple small angle scattering of an electron from stationary ions in a plasma is:

$$\tau_e = \frac{\pi^3\varepsilon_0^2 m_e^2 v_e^3}{2Z^2 e^4 n_i \ln\Lambda} .$$

[8 marks]

(iii) What is $\ln\Lambda$? How is it usually calculated? Show that the resistivity of a plasma is;

$$\eta \propto Z \ln\Lambda\,(k_B T_e)^{-3/2} .$$

Assume quasi-neutrality.

[5 marks]

(iv) Discuss qualitatively how the resistivity of a plasma is modified if a magnetic field is present.

[3 marks]

[TOTAL 20 marks]

4. (i) What is a tokamak? Why is a toroidal current necessary for plasma confinement in a tokamak? How do particle drifts affect the orbits of trapped particles in a tokamak magnetic field configuration? Show the shape of trapped and untrapped particle orbits in a tokamak (including the Larmor motion). [10 marks]

(ii) The total magnetic field in a tokamak can be approximated by;

$$B = B_0 \left(1 - \frac{r}{R} \cos\theta\right)$$

where r is the minor radius of the tokamak and R is the major radius and θ is the poloidal angle such that $\theta = 0$ is at the outside of the torus (where B is a minimum).

Show that the trapping condition is;

$$\frac{v_{\parallel 0}}{v_{\perp 0}} < \frac{\sqrt{2}}{a^{1/2}}$$

$$\text{for } a \gg 1$$

where $a = R/r =$ aspect ratio, $v_{\parallel 0}$ is the velocity parallel to the magnetic field at $\theta = 0$, and $v_{\perp 0}$ is the velocity perpendicular to the magnetic field at $\theta = 0$. [5 marks]

(iii) Considering only the curvature drift for the motion of an electron which is just barely trapped, estimate the width of a banana orbit in a tokamak in terms of a, $v_{\parallel}$, and B_p (the poloidal magnetic field). [5 marks]

[TOTAL 20 marks]

5. (i) What are flux surfaces? Show that in ideal MHD magnetic field lines lie on a flux surface in equilibrium.

[6 marks]

(ii) Show that in ideal MHD

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

where **u** is the fluid velocity.
How is this equation changed if resistivity is included?

[5 marks]

(iii) Show that magnetic field lines are "frozen" in the plasma in ideal MHD.
What occurs if resistivity is non-negligible?

[9 marks]

[TOTAL 20 marks]

Note that;

$\nabla(\mathbf{A}\cdot\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$

and;

$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla \times \nabla \times \mathbf{A}.$

6. Write short notes on THREE of the following topics (using equations and diagrams to illustrate).

(i) Ignition

(ii) Alfvén waves

(iii) Polarisation drift

(iv) Plasma parameter

(v) Bump-on-tail instability

(vi) Critical density

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

QUANTUM OPTICS

For Third- and Fourth-Year Physics Students

Tuesday 27th May 2003: 14.00 to 16.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) What are the main features of the Black body spectrum? [3 marks]

(ii) The probability of there being n photons in the mode with frequency ω inside a black body at temperature T is given by the Boltzmann formula. The density of states of a photon gas is given by

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3}$$

Show that the Planck black body formula is given by the following expression

$$W_T = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}$$

Identify all the terms in the above formula. [7 marks]

(iii) (a) Calculate the standard deviation in the photon number in terms of the average number of photons $\langle n \rangle$. [6 marks]

(b) Which term in the expression of the standard deviation carries the signature of particle behaviour and which reflects wave behaviour and why (a qualitative answer is sufficient here)? [4 marks]

You may use the following expression:

$$\frac{\sum_{n=0}^{\infty} n e^{-n\hbar\omega/kT}}{\sum_{m=0}^{\infty} e^{-m\hbar\omega/kT}} = \frac{1}{e^{\hbar\omega/kT} - 1}$$

[TOTAL 20 marks]

2. Describe briefly the main features of laser light.

[3 marks]

(i) Imagine a one dimensional vacuum cavity with perfectly reflecting mirrors at $x = 0$ and $x = L$. Write down the expression for the wavelengths and frequencies of the allowed modes of the cavity as a function of L?

[4 marks]

(ii) Suppose that each mode i can be represented as a wave of the form

$$E_i = E_0 e^{i\omega_i t}$$

where the amplitude E_0 is the same for all modes, and ω_i is the frequency of the i-th mode. Write down the expression for the total intensity of light inside the cavity taking into account N modes? At what times do the intensity peaks occur and what is the width of each pulse?

[5 marks]

(iii) A more realistic form of the ith mode amplitude is

$$E_i = E_0 e^{i(\omega_i t + \delta_i)}$$

where δ_i is the extra phase of the ith mode. Comment on the physical origin of this extra phase. What is the total intensity as the number of modes becomes very large assuming that the extra phases, δ_i, are completely randomly distributed?

[5 marks]

(iv) How would you make very short laser pulses with a very high intensity?

[3 marks]

[TOTAL 20 marks]

3. (i) Explain briefly the semi-classical approximation in the treatment of light-matter interactions.

[3 marks]

(ii) A nuclear spin has two possible states in an external magnetic field, up $|\uparrow\rangle$ (i.e. aligned with the field) and down $|\downarrow\rangle$ (i.e. anti-aligned with the field). Suppose that the nucleus is in an external (static) magnetic field of strength B, which points in the z direction.

(a) Write down the Hamiltonian for the nucleus using the Pauli matrix notation and identify its eigenvalues.

[4 marks]

(b) Suppose that the initial state of the system is aligned with the field in the z direction, $|\uparrow\rangle$. Suppose then that the field is instantaneously switched to the x direction. Solve the Schrödinger equation to obtain the exact evolution of the nuclear spin in terms of the eigenstate of the Pauli spin matrix σ_x. What is the phase difference between the two orthogonal spin eigenstates of σ_x as a function of time?

[6 marks]

(c) After what time will the spin switch to *its* orthogonal state $|\downarrow\rangle$?

[4 marks]

(iii) What is the relationship between the energy associated with the spin and the time it takes to evolve between orthogonal states? Comment on the validity of the time-energy uncertainty relation to estimate the time of this transition.

[3 marks]

The Pauli matrices are given in the $|\uparrow\rangle, |\downarrow\rangle$ basis by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[TOTAL 20 marks]

4. A quantum particle moving non-relativistically in one dimension has mass m and potential energy $\frac{1}{2}m\omega^2 x^2$. Write down its Hamiltonian H. Express H in terms of the operators

$$\begin{aligned} a &= \frac{\beta}{\sqrt{2}}(x + i\frac{p}{m\omega}) \\ a^\dagger &= \frac{\beta}{\sqrt{2}}(x - i\frac{p}{m\omega}) \end{aligned}$$

where $\beta^2 = m\omega/\hbar$. You may assume that $[x, p] = i\hbar$.

[4 marks]

(i) Evaluate the commutators $[a, a^\dagger]$, $[H, a^\dagger]$ and $[H, a]$.

[4 marks]

(ii) Hence determine the allowed energy levels of the particle, explaining carefully the logic that you use. What do these levels represent when we apply them to a single mode of the quantized electro-magnetic field?

[7 marks]

(iii) Let $|0\rangle$ denote the ground state. Show that

$$\begin{aligned} \langle 0|(a + a^\dagger)|0\rangle &= 0 \\ \langle 0|(a + a^\dagger)^2|0\rangle &= 1 \end{aligned}$$

What do these relationships signify in relation to the quantized electro-magnetic field?

[5 marks]

[TOTAL 20 marks]

5. An atom has two energy levels, i and j, separated in energy by $\hbar\omega_{ij}$. It is subject to a small external time dependent monochromatic electro-magnetic perturbation for a time T, oscillating at the frequency ω. You may assume that the perturbation has matrix elements $V_{ji} = V_{ij}^*$ between these states. Show that if the atom is initially in the state i, the probability of a transition to the state j is approximately

$$P_{ij} = 4|V_{ij}|^2 \frac{\sin^2((\omega_{ij} - \omega)T/2)}{(\hbar(\omega_{ij} - \omega))^2}$$

[10 marks]

(i) Argue that the probability of the transition back from j to i is the same as that for the transition from i to j.

[3 marks]

(ii) Show that, within the formalism employed, the transition rate grows linearly with time.

[3 marks]

(iii) Why is the Einstein B coefficient independent of time (only a qualitative explanation required)?

[4 marks]

[TOTAL 20 marks]

6. Discuss briefly THREE of the following:

(i) Stimulated emission.

(ii) The zero-point energy.

(iii) The classical model of the light-matter interaction.

(iv) Rabi Oscillations.

(v) Phase Matching.

(vi) Coherent States.

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

SPACE PHYSICS

For Third- and Fourth-Year Physics Students

Wednesday 21st May 2003: 14.00 to 16.00

Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) An important equation in space plasma physics is Ohm's law:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.1}$$

where $\mathbf{j}$ is the current density vector, σ the electrical conductivity, $\mathbf{v}$ the plasma velocity and $\mathbf{E}$, $\mathbf{B}$ the electric and magnetic fields respectively. Using Ohm's law, describe a common assumption which can be made in space plasmas by which one is able to show that the electric field arises purely from plasma motion.

[2 marks]

(ii) Write down the magnetic induction equation which describes the evolution of the magnetic field with time. Describe each of the terms in this equation, as well as what happens in the limit of high conductivity.

[4 marks]

(iii) Using your answers from (i) and (ii), describe under what conditions the magnetohydrodynamic (MHD) approximation holds. What is the consequence of this approximation and give two examples of where it breaks down.

[3 marks]

(iv) Use Ampere's law to show that the $\mathbf{j} \times \mathbf{B}$ force can be written as:

$$\mathbf{j} \times \mathbf{B} = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla\mathbf{B}) \tag{1.2}$$

where μ_0 is the plasma permeability. Describe the nature of each term on the right hand side of the equation (1.2) and their effect on the plasma.

[5 marks]

Make use of the following vector identity as necessary, where $\mathbf{G}$ and $\mathbf{F}$ are arbitrary vectors.

$$(\mathbf{G}\cdot\nabla)\mathbf{F} = \frac{\mathbf{1}}{\mathbf{2}}[\nabla\times(\mathbf{F}\times\mathbf{G}) + \nabla(\mathbf{F}\cdot\mathbf{G}) - \mathbf{F}(\nabla\cdot\mathbf{G}) + \mathbf{G}(\nabla\cdot\mathbf{F}) - \mathbf{F}\times(\nabla\times\mathbf{G}) - \mathbf{G}\times(\nabla\times\mathbf{F})]$$

(v) Describe the concept of the Debye length, λ_D, of a plasma and quasi-neutrality. Show that from:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) = \frac{-en_o}{\varepsilon_o}\left(1 - \exp\left(\frac{e\Phi(r)}{kT}\right)\right)$$

that for sufficiently high temperatures this can be written as:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) \cong \lambda_D\Phi(r)$$

where e is the electrical charge, n_o the number density, ε_o the permittivity, k is Boltzmann's constant and T the temperature. Derive the form of λ_D .

[4 marks]

(vi) Using the form of λ_D derived in (v), a numerical expression for the Debye length can be derived which is given by:

$$\lambda_D = 69\sqrt{\frac{T}{n_o}}$$

where the units of λ_D in this case are metres, T is in units of 10^6K, n_o is in cm^{-3}. Using typical solar wind parameters near 1 astronomical unit of $T \sim 10^5$K and $n_o \sim 10$cm^{-3}, calculate λ_D and decide whether the solar wind plasma in this case is quasi-neutral.

[2 marks]

[TOTAL 20 marks]

2. (i) The radiation belts of the Earth consist of energetic charged particles trapped by the Earth's dipole magnetic field. Such electrically charged particles undergo three different types of motion. List the three types and briefly describe the resulting motion for each. [3 marks]

(ii) The equation of motion of a particle of charge q, mass m and velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, is:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ .$$

Let us assume that $\mathbf{E} = \mathbf{0}$ and that there is a constant homogenous magnetic field $\mathbf{B}$ in the z-direction only, so $\mathbf{B} = (0, 0, \mathrm{B})$. Describe briefly without derivation the resultant particle motion in the (x,y) plane, as well as in three dimensions. [2 marks]

(iii) In the more general case of a constant homogenous $\mathbf{E}$ and $\mathbf{B}$ field, use the equation of motion to show that the velocity of the particle perpendicular to $\mathbf{B}$ can be written as:

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \left(\frac{m}{qB}\right)\frac{d}{dt}\left(\mathbf{v} \times \hat{\mathbf{b}}\right)$$

where $\hat{\mathbf{b}}$ is the unit vector along $\mathbf{B}$. Describe each of the terms on the right hand side of this equation. [6 marks]

(iv) A gradient $\mathbf{B}$ drift velocity arises for charged particles moving in a magnetic field which varies in strength in a direction perpendicular to $\mathbf{B}$, given by:

$$\mathbf{v}_{gd} = \frac{mv_\perp^2}{2qB^3}(\mathbf{B} \times \nabla B)$$

where B=|$\mathbf{B}$|, is the magnetic field magnitude and $\mathbf{v}_\perp$ is the velocity of the particle perpendicular to the magnetic field. We concentrate our attention in the Earth's equatorial plane and use a cylindrical co-ordinate system ($\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$). Then the magnetic field at a distance r from the centre of the Earth is given by:

$$\mathbf{B}_{eq}(r) = B_o\left(\frac{R_E}{r}\right)^3 \hat{\mathbf{z}}$$

where B_o is the field strength at the surface of the Earth at the equator, R_E is the radius of the Earth, and $\hat{\mathbf{z}}$ is the unit vector pointing North. Derive the gradient drift motion which the particles will undergo. Use the following as necessary:

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \ .$$

[3 marks]

(v) Using your answer from (iv) explain the behaviour of the ions and electrons. Describe the current and $\mathbf{B}$ field which result from their motion. Use a simple schematic (looking down on the Earth's equatorial plane) to show the effects of this drift motion. [6 marks]

[TOTAL 20 marks]

3. The magnetic field of the Earth is represented by a dipole field, with the rotation and magnetic axes aligned. In spherical polar co-ordinates the magnetic field components of such a dipole field can be written as:

$$B_r = -\frac{2M\cos\theta}{r^3}, \;\; B_\theta = -\frac{M\sin\theta}{r^3}, \;\; B_\phi = 0$$

where M is the dipole moment of the Earth, r is the radial distance from the center of the planet, and θ is the co-latitude.

(i) Show that the magnitude of the **B** field is given by:

$$B(r,\theta) = B_{eq}\frac{R_E^3}{r^3}\sqrt{1+3\cos^2\theta}$$

where R_E is the radius of the Earth. What is B_{eq}? [2 marks]

(ii) Write down the general vector equation describing a field line, as well as explicitly writing down the resultant field line equations in spherical polar co-ordinates. Use these to show that the equation of a field line in the Earth's dipole magnetic field can be written as:

$$r = r_{eq}\sin^2\theta\,.$$

Define r_{eq}. [6 marks]

(iii) A magnetospheric cavity forms as a result of the interaction between the solar wind plasma and the Earth's magnetic field. Explain briefly and conceptually how and which two basic plasma physics principles can be used to explain the formation of the magnetosphere. [3 marks]

(iv) Under what conditions can magnetic reconnection occur, and give two examples found in the Earth's magnetosphere where reconnection regularly occurs. Use a set of simple schematics to show the reconnection of southward interplanetary magnetic field lines across the magnetopause boundary with planetary field lines, as well as the behaviour of the resulting reconnected field lines. [6 marks]

(v) Describe briefly the conditions necessary for the generation of a magnetospheric substorm, as well as the ordered sequence of events which occurs during such a substorm. [3 marks]

[TOTAL 20 marks]

4. The solar wind is a continuous stream of plasma which flows radially away from the Sun at a velocity v. The differential equation which describes this outflow can be derived from the equations of conservation of mass and momentum and is given by:

$$\left(v^2 - \frac{2k_BT}{m}\right)\frac{1}{v}\frac{dv}{dr} = \frac{4k_BT}{mr} - \frac{GM_S}{r^2} \qquad (4.1)$$

where it has been assumed that the plasma pressure $p = 2nk_BT$, in which k_B is Boltzmann's constant, T the temperature, G the gravitational constant, m the mass of the solar wind particles and M_S the solar mass.

(i) Sketch the behaviour of the unique solution of equation (4.1) which describes the behaviour of the outflow of the solar wind. Show the critical radius and the corresponding velocity (derive values for each). Explain why this unique solution is the solar wind solution. Show also on the sketch the four families of solutions which are possible and briefly discuss the physical admissibility of each.
[10 marks]

(ii) The solar atmosphere rotates about an axis nearly perpendicular to the ecliptic plane with a rotation period of 25.4 days at the equator. Define the angular frequency (ω_S) of the Sun in the equatorial plane. This solar rotation results in the interplanetary magnetic field (IMF) lines taking the shape of a spiral as they move outwards away from the Sun. Draw a simple schematic, in the equatorial plane, which shows the behaviour of a single field line at successive time intervals beginning at an initial solar longitude of ϕ_o (at $t = t_o$) and ending at ϕ at time t.
[2 marks]

(iii) On transforming into a frame of reference rotating with the Sun (so the source of the plasma and field lines remains fixed), derive the Archimedian spiral equation:

$$\phi(r) = \phi_o - \frac{\omega_S}{v_{sw}}(r - r_o)$$

where r is the radial distance away from the Sun (which equals r_o at $t = t_o$), ω_S is the angular frequency of the Sun and the solar wind speed $v = v_{sw}$.
[3 marks]

(iv) The angle made by a field line to the radial direction is called the spiral angle and is given by ψ. Show that

$$\tan\psi = \frac{\omega_S(r - r_o)}{v_{sw}}\ .$$

[3 marks]

(v) For a solar wind speed of $400\,\mathrm{kms}^{-1}$, calculate the value of the spiral angle at Earth orbit, where $1\,\mathrm{AU} = 215\,\mathrm{solar\ radii} = 1.5 \times 10^8\,\mathrm{km}$. Draw a simple schematic of the spiral magnetic field lines in the equatorial plane, including the shape of the field lines at 1AU.
[2 marks]

[TOTAL 20 marks]

5. (i) Two different types of plasma flow arise in the Earth's magnetosphere; corotation and convection. Give a brief description of how each of these flows are formed and describe the resultant flow pattern for each.

[4 marks]

(ii) A simple model of the combined corotation/solar wind driven convection (Dungey cycle) flow may be obtained by taking the electrostatic potential in the equatorial plane to be:

$$\Phi(r, \phi) = -\left[E_o r \sin\phi + \frac{\omega_p B_{eq} R_p^3}{r}\right]$$

where E_o is the cross-magnetospheric electric field associated with the Dungey cycle, ω_p the angular frequency of the planetary rotation, B_{eq} the equatorial field strength at the planet's surface, R_p the planet's radius and ϕ the azimuthal angle measured positive anti-clockwise from the noon meridian towards dusk. The gradient vector in polar co-ordinates in a plane is:

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Use this information to find the position of the equatorial stagnation point (R_{sp}, Φ_{sp}) in the flow (that is, where $\mathbf{E} = 0$). Show that the value of the potential at the stagnation point is $\Phi_{sp} = -2E_o R_{sp}$.

[6 marks]

(iii) The location of the stagnation streamline can be found by writing $\Phi(r, \phi) = \Phi_{sp}$. Show that this equation reduces to a quadratic for (r/R_{sp}), and that the solution to this equation is:

$$\frac{r}{R_{sp}} = \frac{\left(1 \pm \sqrt{1 - \sin\phi}\right)}{\sin\phi}.$$

[3 marks]

(iv) Draw a sketch in the equatorial plane showing the combined flow pattern at the Earth (with dusk and noon clearly marked). Point out the stagnation point, stagnation streamline and comment on the resultant flows and on that which is dominant at Earth.

[7 marks]

[TOTAL 20 marks]

6. A cavity, known as the magnetosphere, forms around the Earth as a result of the interaction between the solar wind and the Earth's magnetic field.

(i) Draw an overview sketch of the Earth's magnetosphere in the noon-midnight meridian plane. Show clearly on the sketch and discuss briefly each of the following:

(a) Solar wind flow and bow shock.

(b) Magnetosheath.

(c) Magnetopause.

(d) Earth's magnetic field lines, on dayside and nightside.

(e) Magnetotail and the tail reconnection point.

[6 marks]

(ii) Seen in cross-section, the near Earth geomagnetic tail consists of two D-shaped lobes of oppositely directed magnetic fields, separated by a current sheet in the centre of the tail. If one stands far downtail and looks back towards the Earth, draw a schematic showing a view of the cross-tail current and tail lobe magnetic field lines (of constant field strength $B_T \sim 20\,\text{nT}$). Also show the magnetopause currents via which the tail current closes.

[3 marks]

(iii) Assuming the diameter of the tail is $\sim 40\,R_E$ (where the Earth's radius $= R_E = 6400\,\text{km}$), and the average cross-tail electric field associated with the solar wind driven convection cycle is $\sim 2 \times 10^{-4}\ \text{Vm}^{-1}$, one is able to estimate that it takes 3.6 hours for an open flux tube to flow from the northern (or southern) tail lobe magnetopause to the central current sheet. This time is also an estimate for the duration that a given field line remains open during the solar wind driven convection cycle, whilst it is being carried downstream away from the Earth by the solar wind at a velocity of $\sim 400\,\text{km}\,\text{s}^{-1}$.

Draw a schematic in the noon-midnight meridian plane showing closed dayside field lines, reconnected open field lines in the cusp region, open field lines flowing downstream and then reconnected field lines in the centre of the magnetotail. Also show the direction of plasma flow in the nightside magnetosphere.

Calculate the length of the geomagnetic tail in R_E.

[6 marks]

(iv) Draw a schematic of the view looking down on the north pole of the Earth showing the form of the plasma flow in the ionosphere which is associated with the solar wind driven convection cycle. Mark the local times of noon, dusk, midnight and dawn on the figure. Show circles representing the open flux and closed flux regions. Complete the diagram by requiring the ionospheric plasma to flow around closed paths.

[5 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

BIOPHYSICS OF NERVE CELLS & NETWORKS
With Advanced Study

For Third - and Fourth -Year Physics Students

Monday 19th May 2003: 14.00 to 17.00

Answer THREE questions from Section A and ONE question from Section B.

Each question from Section A carries half as many marks as those carried by questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

$R = 8.314\ \mathrm{JK^{-1}mol^{-1}}$, $F = 96,400\ \mathrm{C\,mol^{-1}}$
The distributions of sodium, potassium and chloride ions across the membrane of a nerve cell can be taken to be:

Ion	**Inside**	**Outside**
Sodium	14 mM	125 mM
Potassium	124 mM	5 mM
Chloride	6 mM	77 mM

SECTION A

1. (i) Starting from the relationship between the membrane current per unit length and the variation of membrane potential along an unmyelinated nerve axon, show that the current i_m that flows per unit area across the membrane is given by

$$i_m = \frac{a}{2\rho_i \theta^2}\left(\frac{\partial^2 V_m}{\partial t^2}\right)$$

where ρ_i is the resistivity of the axoplasm, a is the axon radius, θ is the velocity of the action potential and V_m is the membrane potential. [12 marks]

(ii) What can be inferred from the equation in part (i) about how the velocity of the action potential in an unmyelinated nerve varies with axon diameter? [3 marks]

(iii) What is meant by the "refractory period"and how does it affect the properties of the nerve axon? [2 marks]

(iv) What is myelin and how does it affect the propagation of nerve action potentials? [3 marks]

[TOTAL 20 marks]

2. (i) Explain the principle behind the voltage-clamp method for studying the electrical properties of nerve membranes.

[4 marks]

(ii) How was this method employed to describe the time-course of the changes in sodium ion conductance in the membrane of the squid giant axon following a sudden depolarisation?

[6 marks]

(iii) Hodgkin and Huxley described the sodium current across the axonal membrane in terms of the following equation

$$I_{Na} = m^3 h \bar{g}_{Na} \left(V_m - V_{Na}\right).$$

What do the various parameters in this equation represent and what are the main assumptions made in its derivation?

Show that the time course of h is given by an equation of the form:

$$h = h_\infty - (h_\infty - h_0) \exp\left(\frac{-t}{\tau_h}\right).$$

[6 marks]

(iv) Sketch the time-course of the parameter h under the following conditions

(a) The axon has been held under voltage clamp for a prolonged period at $-100\,\mathrm{mV}$ and is then stepped to $0\,\mathrm{mV}$.

[2 marks]

(b) The axon has been held under voltage clamp for a prolonged period at $0\,\mathrm{mV}$ and is then stepped to $-100\,\mathrm{mV}$.

[2 marks]

[TOTAL 20 marks]

3. (i) From first principles, show that the difference in electrochemical potential across a membrane can be written

$$\Delta\mu = RT \ln\left(\frac{c_{in}}{c_{out}}\right) + zFV_{mem}\,.$$

[7 marks]

(ii) From this equation derive the Nernst equation and show, using the ionic concentration data given in the Table, that a nerve cell cannot be at thermodynamic equilibrium. [5 marks]

(iii) Consider a neuron under voltage-clamp held at a membrane potential of $-40\,\mathrm{mV}$. Assuming the cell expressed both $\mathrm{GABA_A}$ receptors and nicotinic acetylcholine receptors, would inward or outward currents be observed when each of these neurotransmitters was applied to the cell? Explain your reasoning.

[4 marks]

(iv) In a voltage clamp experiment, how would you determine if these currents would be excitatory or inhibitory in the un-clamped neuron? [6 marks]

[TOTAL 20 marks]

4. (i) Describe the basic mechanisms underlying the transmission of information across a chemical synapse. How do chemical synapses differ from electrical synapses?

[5 marks]

(ii) Describe two experiments that would show that calcium must move across the presynaptic membrane in order to trigger neurotransmitter release.

[5 marks]

(iii) What evidence suggests that transmitter release is quantal at the neuromuscular junction?

[5 marks]

(iv) Outline how the patch clamp technique can be used to study the properties of postsynaptic neurotransmitter receptors. Under what circumstances might noise analysis be used instead of patch-clamping?

[5 marks]

[TOTAL 20 marks]

5. Are the following statements true or false? Explain your reasoning (N.B. no reasoning, no marks).

(i) The height of an action potential is affected by external sodium ion concentrations but this has little effect on the resting membrane potential. [2 marks]

(ii) For myelinated nerves, the ratio of the internal to the external diameter is roughly constant. [2 marks]

(iii) Ion flow down an open voltage-gated sodium channel does not require energy. [2 marks]

(iv) Sustained action potentials result in the consumption of energy. [2 marks]

(v) The breakdown field for nerve membranes is $50,000\,\mathrm{V\,cm^{-1}}$. [2 marks]

(vi) The time-course of an excitatory postsynaptic current is determined by the diffusion of the neurotransmitter away from the postsynaptic membrane. [2 marks]

(vii) A positively charged local anaesthetic such as lidocaine binds tighter to its target at depolarised potentials. [2 marks]

(viii) Rod cells in the retina depolarise when exposed to light. [2 marks]

(ix) Visual acuity at night (as opposed to during the day) is enhanced at the expense of sensitivity. [2 marks]

(x) Colour blindness is usually a consequence of a vitamin deficiency. [2 marks]

[TOTAL 20 marks]

6. Write notes on ALL THREE of the following:

(i) The measurement of the Hodgkin-Huxley parameters h_∞ and τ_h

(ii) The role of the sodium pump in neuronal excitability.

(iii) The role of horizontal cells in contrast enhancement in the retina.

[20 marks]

SECTION B

7. Write an essay on the content and significance of ONE of the following papers:

(i) "Identification of a cold receptor reveals a general role for TRP channels in thermosensation" McKemy et al. Nature 416, 52–58 (2002).

(ii) "Neurotrophin-evoked depolarization requires the sodium channel Nav1.9" Blum et al. Nature 419, 687–693 (2002).

(iii) "The sedative component of anesthesia is mediated by GABAA receptors in an endogenous sleep pathway" Nelson et al. Nature Neuroscience 5, 979–984 (2002).

[40 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

COSMOLOGY
With Advanced Study

For Fourth-Year Physics Students

Wednesday 28th May 2003: 14.00 to 17.00

Answer THREE questions from Section A and TWO question from Section B.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

SECTION A

1. (i) By applying the adiabatic form of the first law of thermodynamics, $\mathrm{d}E = -p\mathrm{d}V$, to an element of the substratum, derive the fluid equation. [3 marks]

(ii) In a Universe dominated by radiation the pressure is given by $p_r = \rho_r c^2/3$. Show that the solution to the fluid equation is:

$$\rho(t) = \rho_0 \left(R(t)/R_0\right)^{-4}$$

where ρ_0 and R_0 are the density and scale factor at the present time. [4 marks]

(iii) What is the corresponding solution for a matter dominated universe? [3 marks]

(iv) The Friedmann equations (for $\Lambda = 0$) are:

$$\begin{aligned} \ddot{R} &= -\frac{4\pi G\rho}{3}R \\ \dot{R}^2 &= \frac{8\pi G\rho}{3}R^2 - kc^2. \end{aligned}$$

Show from these and the fluid equation that the expansion of the early Universe is characterized by the expression $R(t) \propto t^{1/2}$. [4 marks]

(v) Assuming $k = 0$, the corresponding solution for a matter dominated Universe is $R(t) \propto t^{2/3}$. Estimate from this the time when matter and radiation were equally important constituents of the universe. [6 marks]

[TOTAL 20 marks]

[Assume the ratio of the present densities of radiation and matter to be $\rho_{r,0}/\rho_{m,0} = 3.5 \times 10^{-5}$ and the age of the Universe $t_0 = 14\,\mathrm{Gyr}$]

2. (i) The intensity of blackbody radiation is given by the Planck spectrum:

$$I(\nu)d\nu = B_\nu(T)d\nu = \left(\frac{2h}{c^2}\right)\left(\frac{\nu^3}{e^{h\nu/k_BT}-1}\right)d\nu \qquad (2.1)$$

Where ν is the frequency, h is Planck's constant, k_B is Boltzmann's constant, and T is the temperature. Show that the total radiation density is given by the expression $u_{rad} = aT^4$, where a is a constant.

[4 marks]

(ii) The Cosmic Microwave Background (CMB) has $T = 2.73\,\mathrm{K}$. Evaluate the density parameter of the CMB, Ω_{rad}, at the present epoch.

[4 marks]

(iii) Show from equation 2.1 that the CMB retains its blackbody form during the expansion of the universe, and estimate the temperature and energy density at the time of recombination at redshift $z \sim 1000$.

[4 marks]

(iv) Observations of CMB fluctuations suggest that the current total density parameter of the Universe Ω_{tot} is very close to 1. Discuss the major contributors to the density of the Universe (in addition to radiation) and their current importance.

[8 marks]

[TOTAL 20 marks]

[You may use the following:
Useful formula $\displaystyle\int_0^\infty \frac{y^3}{e^y-1}dy = \frac{\pi^4}{15}$
Gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-1}}$
Hubble constant $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$
1 pc = $3.09 \times 10^{16}\,\mathrm{m}$
Speed of light $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$
Boltzmann constant $k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$]

3. (i) Discuss the main lines of evidence supporting the hot big bang theory. Illustrate your answer with equations or diagrams, as necessary.

[9 marks]

(ii) For an Einstein de Sitter model ($k = 0$, $\Lambda = 0$, $p = 0$), determine the age of the Universe, t_0, in years for a Hubble constant $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and discuss the consistency of this with other estimates.

[4 marks]

(iii) Observations suggest our own Universe has $k = 0$, and that the density parameter associated with the cosmological constant, $\Omega_\Lambda = 0.7$. Using the Friedmann equation (5.1), show that the age of the universe is given by:

$$t_0 = \int_0^{t_0} dt = \frac{1}{H_0}\int_0^1 \left(\frac{1-\Omega_\Lambda}{x} + \Omega_\Lambda x^2\right)^{-1/2} dx, \text{ where } x = \frac{R}{R_0}.$$

This has the solution:

$$t_0 = \frac{2}{3H_0}\Omega_\Lambda^{-1/2}\ \ln\left(\frac{1+\Omega_\Lambda^{1/2}}{(1-\Omega_\Lambda)^{1/2}}\right).$$

For the Hubble constant above, evaluate the age of the Universe in this case and comment also on the consistency of this result.

[7 marks]

[TOTAL 20 marks]

[$1\,\mathrm{pc} = 3.09 \times 10^{16}\,\mathrm{m}$]

4. (i) Explain what is meant by cosmological inflation, and describe how this may solve three problems associated with the standard big bang model.
[7 marks]

(ii) Some cosmological models predict that the Universe expands as $R \propto t^m$ where m is an arbitrary constant. What range of values of m corresponds to a satisfactory inflationary expansion?
[4 marks]

(iii) Magnetic monopoles behave as non-relativistic matter. Suppose that at a temperature corresponding to the Grand Unified era, about 3×10^{28} K, magnetic monopoles were created with a density of $\Omega_{mon} = 10^{-10}$. Assuming that the Universe has a critical density and is radiation dominated, what would the temperature be when the density of monopoles equals that of radiation?
[4 marks]

(iv) In the present Universe, $T \approx 3$ K. Compute the value $\Omega_{mon}/\Omega_{rad}$ would have at the present day. Assuming $\Omega_{rad} = 10^{-4}$, is this ratio compatible with observations? If the radiation density stays constant during inflation, how much inflationary expansion is necessary so that the present-day density of monopoles matches that of radiation?
[5 marks]

[TOTAL 20 marks]

5. (i) In General Relativity, the Friedmann equation is:

$$\dot{R}^2 = \frac{8\pi G\rho}{3}R^2 - kc^2 + \frac{\Lambda}{3}R^2 \tag{5.1}$$

where R is the scale factor of the universe, G is the gravitational constant, ρ is the density of matter, c is the speed of light and k is a constant. For $\Lambda = 0$, discuss the solution, $R(t)$, of the Friedmann equation in a matter-dominated universe for the following cases:

(a) Where the mass of matter in the Universe is negligible

(b) Where the mass is non-negligible and $k = 0$

(c) Where the mass is non-negligible and $k = -1$

(d) Where the mass is non-negligible and k = 1

Illustrate your answers with a graph of $R(t)$ in each case. [10 marks]

(ii) What is the meaning of the term Λ in Eq. 5.1? Give examples of the interpretation of this term and briefly describe the evidence that, at the current epoch, $\Lambda > 0$. [4 marks]

(iii) For a universe with $\Lambda > 0$ and $k \leq 0$, describe the behaviour of the scale factor $R(t)$ and derive the late-time behaviour of $R(t)$ in this case. Discuss the significance for our own Universe.

[6 marks]

[TOTAL 20 marks]

6. (i) What is the meaning of the term k in the Robertson-Walker Metric:

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right).$$

State why k can take values of -1, 0 and $+1$ without losing generality, and describe Universes characterized by these values.

[4 marks]

(ii) In an expanding universe described by the Robertson-Walker metric, by considering photons moving radially towards us from a distant observer, show that the light signal is redshifted by a factor:

$$1 + z = \frac{R_{t_0}}{R_{t_e}}$$

where t_0 and t_e are the times at which the photon is observed and emitted respectively.

[6 marks]

(iii) The most distant object currently known is a quasar at $z = 6.43$. Assuming a matter-dominated Universe with $k = 0$,

(a) How much more dense was the universe at this epoch, compared to the present one, and what was its temperature?

[2 marks]

(b) What is the proper distance to the quasar?

[5 marks]

(iv) What observational and theoretical factors limit our ability to see objects at even greater distances?

[3 marks]

[TOTAL 20 marks]

[Hubble constant $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$
$1\mathrm{pc} = 3.09 \times 10^{16}\,\mathrm{m}$]

SECTION B

7. Write an essay on the uses of gravitational lensing in cosmology.

[20 marks]

8. Write an essay on black holes and cosmology.

[20 marks]

9. Write an essay on the instabilities of the big bang and on quintessence.

[20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

LASERS, OPTICS & HOLOGRAPHY

With Advanced Study

For Third - and Fourth - Year Physics Students

Tuesday 20th May 2003: 14.00 to 17.00

Answer THREE questions from Section A and BOTH questions from Section B.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Determine the form of the intensity pattern at a large distance r due to diffraction of a plane wave by the following apertures:

(a) A single narrow slit aperture of width $2a$ (you may assume that this is infinitely long).

(b) A rectangular aperture with sides of lengths $2a$ and $2b$. (You may assume that $r \gg a, b$.)

(c) For a rectangle of sides $150 \times 600\,\mu$m illuminated by plane light at a wavelength of 550 nm at what angle from the optic axis are the first minima found in the two orthogonal directions parallel to the edges of the aperture.

[10 marks]

(ii) A rectangular aperture of sides $100 \times 800\,\mu$m is the "object" in a 4-f optical processing system with large aperture lenses. The processing system uses plane wave coherent light of wavelength 632.8 nm, the lenses of the 4-f system all have focal length of 1 m, and there is a small circular aperture of diameter 4 mm in the Fourier plane that acts as a filter.

(a) What is the highest spatial frequency in the image?

(b) Sketch the effect upon the image of this filter.

(c) What is the effect upon the image of increasing or decreasing the illuminating wavelength?

(d) What filter aperture should be used to ensure that the edges of the image are almost as sharp as those in the original object.

[10 marks]

[TOTAL 20 marks]

2. (i) Write down the Ray Transfer Matrix that represents the action upon rays propagating close to the optic axis of:

(a) Free space propagation over a distance d.

(b) A thin lens of focal length f.

(c) From these results determine the Ray Transfer Matrix for the combination of free-space propagation over a distance d followed by a thin lens of focal length f. [5 marks]

(ii) An optical cavity comprises a pair of mirrors (of radii of curvature R_1 and R_2) which are separated by a distance d. We define the parameters g_1 and g_2 as:

$$g_1 = 1 - d/R_1 \quad and \quad g_2 = 1 - d/R_2.$$

It can be shown that for stable solutions only certain values of the product of these parameter $g_1 g_2$ are permitted.

(a) State the constraints on the value of $g_1 g_2$.

(b) Plot in a graph of g_1 versus g_2 the regions where the cavity will be stable.

(c) Which of the following cavities are stable, marginally stable and unstable:

I. $R_1 = -4\,\text{m}$, $R_2 = \infty$, $d = 1\,\text{m}$

II. $R_1 = R_2 = \infty$, $d = 40\,\text{cm}$

III. $R_1 = 0.5\,\text{m}$, $R_2 = 1\,\text{m}$, $d = 0.5\,\text{m}$

IV. $R_1 = 50\,\text{cm}$, $R_2 = 75\,\text{cm}$, $d = 1\,\text{m}$

Plot their positions on the stability diagram. [10 marks]

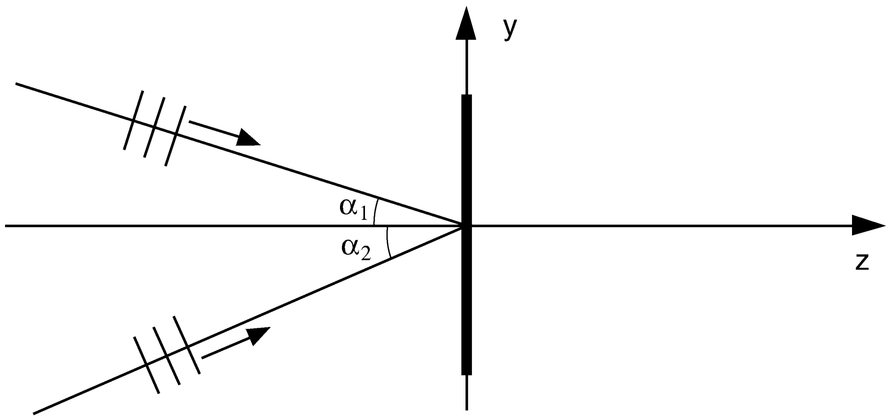
(iii) Consider a laser operating at 1064 nm with a stable cavity of length 1.2 m for which the lowest order laser mode has a minimum waist of 1.3 mm diameter at the centre point of the cavity. There is an output coupling mirror of reflectivity 0.95 (assume the other mirror has reflectivity 1.0).

(a) What is the diameter of the beam waist 2m beyond the output coupler?

(b) Estimate by what factor the intensity 2 m beyond the output coupler will be reduced relative to the intensity at the minimum waist in the cavity? [5 marks]

[TOTAL 20 marks]

3. (i) Two coherent light beams with the same intensity and of wavelength λ are incident upon a recording screen at $(y, 0)$ as shown in the figure.



(a) Show that the intensity at the screen is given by:

$$I(y, 0) = I_o \cos^2\left(\frac{\pi y}{\lambda}(\sin\alpha_1 + \sin\alpha_2)\right)$$

where I_o is a constant.

(b) If $\alpha_1 = \alpha_2 = \alpha$, what value of 2α gives fringes in the screen separated by $1\,\mu$m for illumination with 630 nm wavelength light?

[10 marks]

(ii) (a) Briefly describe the steps required to record, develop and reconstruct a hologram.

(b) Consider the "hologram" formed for the case outlined above in (i)(a) (think of beam 1 as the reference and beam 2 as the object). Determine the diffraction from the hologram. By considering the properties of the different order of diffraction ($m = 0, +1$and -1), identify the diffracted beam that corresponds to the original "object" beam.

[8 marks]

(iii) Outline how holography can be used to measure small motions of a moving surface.

[2 marks]

[TOTAL 20 marks]

4. (i) (a) The polarisation P of a dielectric medium can be expressed in terms of a power series expansion in the electric field strength E of the applied field:

$$P = \varepsilon_0 \left(\chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \ldots\right) . \tag{4.1}$$

This is referred to as the polarisation expansion.
Under what conditions will the first order term dominate?

(b) By considering the medium as an ensemble with a number density N of bound electronic charges, each acting as an oscillator of natural frequency ω_0, the linear response of the medium to an electromagnetic field ($E(t)$ at frequency ω) can be determined in terms of the classical displacement of the charges by the field. This treatment yields

$$P = eNx(t) = \frac{e^2/m_e}{\left(\omega_0^2 - \omega^2\right)} NE(t) . \tag{4.2}$$

Given that the linear susceptibility of the medium is related to the refractive index n by the expression $n^2 = 1 + \chi^{(1)}$ determine an expression that gives the frequency dependence of n^2.

(c) Generalise equation 4.2 to the case of a real medium of atoms which have many resonances each with an oscillator strength f_j and a radiative damping γ_j. [7 marks]

(ii) (a) Sketch the refractive index dependence upon the electromagnetic field frequency for the case of a typical atomic gas indicating where the dipersion is "normal" (i.e. increases with frequency) and where the dispersion is "anomalous" (i.e. decreases with frequency).

(b) By employing a modified version of the model used above (part (i)(b)) briefly explain the phenomenon of birefringence. [5 marks]

(iii) For the high intensities available in a laser field the second order term in the polarisation expansion (equation 4.1) may become important.

(a) What is the condition on the medium symmetry for there to be a non-vanishing second order contribution?

(b) Show that in this circumstance if a field at frequency ω is applied that the second harmonic will be generated.

(c) What constraint is placed upon the refractive index of the medium by the need to conserve momentum in this process?

(d) How can this constraint upon the refractive index be satisfied in a birefringent crystal? [8 marks]

[TOTAL 20 marks]

5. (i) (a) Briefly describe the energy level scheme pertinent to laser action in the Nd:YAG laser (use a sketch if required).

(b) What type of light sources are used for pumping inversion in this system for pulsed or CW operation?

(c) Describe the temporal behaviour of light emission from a simple Nd:YAG laser cavity comprising just mirrors and the gain medium in the case where the pump comprises a pulsed light source capable of pumping inversion in the Nd:YAG crystal by a factor of many times that required for reaching the threshold in the cavity that contains no additional intra-cavity elements.

[8 marks]

(ii) (a) Briefly explain how the output pulse from a Nd:YAG laser can be controlled so as to maximise the output pulse peak power. Identify any intra-cavity optical elements are required to achieve this.

(b) For a cavity of length 250 mm and effective mirror reflectivity $R = 0.92$ containing a Nd:YAG crystal in which a total energy of 0.75 J is stored, what will be the duration and peak power of the output pulse?

[8 marks]

(iii) The second-harmonic of a Q-switched Nd:YAG laser is used to pump a ring laser cavity containing a titanium sapphire crystal and no other intra-cavity elements. This titanium sapphire laser is to be used for the generation of pulses of single mode light.

(a) Explain how single mode operation can be attained in such a cavity if a low power narrow-bandwidth CW laser is also available.

(b) The pulse length of this cavity is typically >5 times longer than the pump laser used, estimate the minimum bandwidth of the output of the titanium sapphire ring laser.

[4 marks]

[TOTAL 20 marks]

6. Write short notes on THREE of the following. Use sketches and state important mathematical expressions where necessary.

(i) The resolution limits to human vision.

(ii) The longitudinal modes of a Fabry-Perot etalon.

(iii) Active and passive laser mode-locking techniques.

(iv) The physical origin of non-linear refractive index.

(v) External cavity stabilisation of a laser diode.

(vi) Phase contrast microscopy.

[TOTAL 20 marks]

SECTION B

7. Answer only ONE part, either (I), (II) or (III).

(I)

(i) An interferometric system is to be used to create an in-fibre Bragg grating with a peak reflection at a wavelength of 1.515 μm (the effective refractive index of the fibre at this wavelength is 1.460). The system uses a frequency quadrupled Nd:Yag laser operating at a wavelength of 244nm.

(a) Sketch the experimental setup and describe the operation of the system. Take care to comment on the key characteristics of the components (including the laser) and their arrangement. [5 marks]

(b) Calculate the required geometry of the interfering beams to achieve peak reflection at the specified wavelength. [4 marks]

(c) Sketch a graph of reflection against wavelength, for wavelengths close to the design wavelength. Discuss the effect of the length of the grating on its spectral characteristics. [3 marks]

(d) Describe two other possible techniques to make fibre Bragg gratings. [2 marks]

(ii) The grating created in part (i) is to be used in an add-drop filter in an optical communications link.

(a) Discuss briefly how the grating may be used to construct an add-drop filter. Sketch a suitable experimental arrangement. [3 marks]

(b) Comment on the issues that are important in assessing the performance of add-drop filters. [3 marks]

[TOTAL 20 marks]

(II)

(i) One popular technique for locking a laser to a Fabry-Perot cavity generates an error signal proportional the differential of the Fabry-Perot cavity lineshape function. You may assume the lineshape of the cavity mode is Lorentzian. Sketch the resulting error signal indicating the positions of its extrema. Comment on why its shape is appropriate for laser locking. Describe one advantage and one disadvantage of using a high-finesse cavity for such a locking system.

[11 marks]

The (un-normalised) Lorentzian function is given by

$$I(\omega) = I_0 \frac{(\Delta\omega/2)^2}{(\omega - \omega_0)^2 + (\Delta\omega/2)^2}$$

(ii) The linewidth of a continuous wave laser may be estimated by sending its output through a Fabry-Perot cavity and monitoring the throughput of the cavity on a photodiode. The laser frequency ω_L is tuned to $\omega_0 + \Delta\omega/2$ where ω_0 is a cavity resonance and $\Delta\omega$ is the full width at half maximum of the cavity mode. In this way, frequency fluctuations of the laser about ω_L generate amplitude fluctuations on the detector. For a particular experiment the maximum throughput of the cavity generates a signal of 1V. The amplitude fluctuations (noise) on the signal is 5mV when $\omega_L = \omega_0$. The noise increases to 150mV at $\omega_L = \omega_0 + \Delta\omega/2$. Estimate the linewidth $\delta\omega$ of the laser. For the purposes of your estimate you may assume the lineshape function to be approximately linear at $\omega = \omega_0 + \Delta\omega/2$. The cavity is of length $l = 15$cm and the finesse of the cavity is 100.

[9 marks]

[TOTAL 20 marks]

(III)

This question is concerned with an attosecond light source based on high harmonic generation. An intense laser pulse of 1 mJ energy interacts with a gas target to produce an attosecond burst of soft x-rays in the range 10–20 nm with an conversion efficiency (= soft-x ray energy/ laser energy) of 10^{-9} into this 10 nm band. A thin foil filter is used to filter out the laser light with a transmission of 0.5 for the soft x-rays (assume constant) and 0.0 for the laser light. The soft x-rays are then incident on a multi-layer x-ray focusing mirror that has a reflectivity of 0.6 for the soft x-rays (assume constant) and a reflection bandwidth of 1 nm (FWHM) at a wavelength of 13 nm.

(a) What is the minimum pulse duration, Δt, for the soft x-rays after reflection from the multi-layer mirror, assuming a transform-limited pulse.

[10 marks]

(b) Hence, calculate the maximum the photon density (photon/m^3) at the focus of the multi-layer mirror, assuming the mirror has a focal length of 35 mm and that the soft-rays are incident as a Gaussian beam of radius $w = 1$ mm (e^{-2} in intensity).

[10 marks]

[TOTAL 20 marks]

8. Answer only ONE part, either (I), (II) or (III).

(I) Write an essay entitled “Sensor applications of fibre Bragg gratings”

[20 marks]

(II) Write brief notes on ALL of the following

(a) Saturated absorption spectroscopy.

(b) The Pound-Drever-Hall laser locking scheme.

(c) Applications of ultra-stable lasers.

[20 marks]

(III) Write notes, including diagrams if necessary, on THREE of the following topics:

(a) Attosecond pulse production using high harmonic generation.

(b) Limitations of autocorrelation techniques for the measurement of soft x-ray attosecond pulses.

(c) Methods for measuring the duration of attosecond pulses.

(d) Applications of attosecond pulses.

[20 marks]

End

UNIVERSITY OF LONDON

MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

OPTICAL COMMUNICATIONS PHYSICS
With Advanced Study

For Fourth-Year Physics Students

Tuesday 3rd June 2003: 14.00 to 17.00

Answer ALL parts of Section A, TWO questions from Section B and TWO questions from Section C.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. State Shannon's formula for channel capacity, clearly defining the terms used and the assumptions made for its validity.

[2 marks]

A 3 minute analogue music clip having frequencies from 200 Hz to 15000 Hz and with a signal to noise power ratio of 40 dB undergoes analogue to digital conversion.

(i) What is the *minimum* number of bits of information in the digital signal required to encode the information of the analogue signal?

[4 marks]

(ii) How long would it take to transmit this information as a binary digital code in a medium with a bandwidth of 10 kHz? How can real-time digital transmission be achieved with this transmission medium?

[4 marks]

(iii) Explain what is meant by additive white Gaussian noise (AWGN).

[2 marks]

(iv) With the aid of suitable diagram(s) describe how the bit error rate of a digital signal due to AWGN can be quantified. [Detailed mathematical calculation is not required].

[4 marks]

(v) Describe the two main types of copper cabling for communication links and explain why the attenuation of the copper increases at high frequencies.

[4 marks]

[TOTAL 20 marks]

SECTION B

2. (i) Explain the problems and solutions to how a mobile telecommunications network handles:

(a) Limited network bandwidth. [4 marks]

(b) Signal degradation associated with free-space signalling. [3 marks]

(ii) Describe, in principle, how a baseband digital signal with frequencies from 0 – 25 kHz can be formatted for free-space communications at central frequency $\nu_0 = 900\,\text{MHz}$. In your answer you should provide a mathematical description of the method in the time and frequency domains and provide sketches of the baseband and mobile signals in the frequency domain with clear scales on the frequency axis. [7 marks]

(iii) A base station in the mobile network emits a signal power of 1 kW uniformly in all directions. A piece of mobile equipment has an effective collection area of $5\,\text{cm}^2$ and requires a power of $1\,\mu\text{W}$ for good reception. What is the maximum range for good reception? [6 marks]

[TOTAL 20 marks]

3. (i) An optical fibre link utilises a narrow bandwidth modulated laser source with an average output power of 1 mW emitting at 1.55 μm. The fibre loss is known to be 0.4 dB/km and the receiver requires a minimum power of -25 dBm in order to achieve an acceptable bit error rate.

(a) Calculate the attenuation-limited length of the link. [2 marks]

(b) Erbium Doped Fibre Amplifiers (EDFAs) might be employed to increase this value. Briefly describe the operation of an EDFA and how this might be incorporated into a system. [4 marks]

(c) Indicate the limitations of such amplifiers for Wavelength Division Multiplexing (WDM) applications. [2 marks]

(d) Briefly outline how dispersion in silica fibres can also limit the length of the link. [3 marks]

(ii) The same fibre is to be assessed for operation at 1.3 μm using the cut- back method to determine the loss at this wavelength. This involves a short length, 2 km, of fibre and an optical receiver. For a particular laser source operating at 1.3 μm the measured output voltage at the receiver was found to be 2.7 V. The fibre was then cut to a length of 2 m and the output voltage increased to 11.2 V. The same source was used in both cases and the receiver is known to be linear in incident optical power. Using this information, calculate the attenuation of the fibre in dB/km at this wavelength. [4 marks]

(iii) Account for the difference between the values of attenuation measured at 1.55 μm and 1.3 μm by outlining the main factors which give rise to signal loss in silica fibres. [5 marks]

[TOTAL 20 marks]

4. (i) Thermal and shot noise in photodiode detectors is frequently described as Gaussian White noise. Explain why this is so by considering the origin of each type of noise. For each case derive expressions for the mean square current.

[6 marks]

(ii) Show that the photocurrent, I_P, generated by a photodiode receiver may be written

$$I_P = \frac{\eta P_0 q \lambda}{hc}$$

where η is the quantum efficiency, q the electronic charge, λ the wavelength of the detected photons and P_0 the received optical power.

[4 marks]

(iii) An InGaAs p-i-n photodiode with a quantum efficiency of 0.7 and a negligible dark current is used to detect $1.3\,\mu$m light. Calculate the output current, I_P, if the average optical power incident on the detector is $-20\,$dBm.

[4 marks]

(iv) Using your expressions derived in part (i) show that the Signal-to- Noise ratio (SNR) for the receiver is given by

$$\mathrm{SNR} = \frac{{I_P}^2}{2q\,(I_P + I_D)\,\Delta f + 4k_B T\,\Delta f / R_L}$$

where T is the ambient temperature, R_L, the load resistor, I_D the dark current and Δf the receiver bandwidth.

[2 marks]

(v) Calculate the shot noise limited SNR (in dB) of the detector described in part (iii) if this detector is used in a 100 Mb/s digital communications link which uses non-return-to-zero coding.

[4 marks]

[TOTAL 20 marks]

5. Briefly discuss THREE of the following: [All parts carry equal marks]

(i) Data compression techniques.

(ii) Dispersion in optical fibres.

(iii) Error detection and correction.

(iv) Optical sources for communications.

(v) Electro-optic modulators for optical switching.

(vi) Packet-switching in networks and its advantages/disadvantages.

[20 marks]

SECTION C

6. (i) Give a comparison of the physical layout and dimensions of data stored on CD and DVD and state the storage capacities of each system. What (approximately) are the diode laser wavelengths in the CD and DVD systems? Explain why the development of a 'blue' diode laser should have impact in the storage capacity of future disc systems. [7 marks]

(ii) Explain how the physical construction of a compact disc (CD) and the use of CIRC-encoding of the information both lead to low error rates when reading data. Describe how the optical read-out system in a compact disc player is able to control focus and tracking on the CD. [7 marks]

(iii) Describe the principle of volume holographic optical storage and discuss the difficulties involved in making this a future high-density storage technology. [6 marks]

[TOTAL 20 marks]

7. The optical network provides the basis for high bandwidth communications on a global and local level. The deployment of the optical network and its capacity is not just determined by technology but also by economical issues.

(i) Comment briefly on the status of the telecommunication industry in the last few years, and name some of the key companies deploying the optical network. [3 marks]

(ii) Describe the services that home customers and businesses currently use and what services might be expected to grow. [4 marks]

(iii) Discuss the technologies and architectures that are being deploying to meet the demand. [6 marks]

(iv) Discuss some of the economics of deploying optical communications in the following parts of the network:

(a) long-haul;

(b) metropolitan;

(c) local area networks; and

(d) fibre-to-the-home.

[7 marks]

[TOTAL 20 marks]

8. (i) Optical fibre amplifiers have become indispensable components in long distance and other high performance optical communications systems. Discuss and compare the various ways in which optical amplification has been realised across the optical communications windows, paying particular attention to $1.3\,\mu$m and $1.55\,\mu$m. Your answer should include references to glass based fibre laser amplifiers, to semiconductor amplifiers and to stimulated Raman amplifiers.

[13 marks]

(ii) As aggregate bit rates approaching 1 Tb/s are being considered for commercial deployment, what are the major challenges that must be addressed in order to realise such high capacity communications systems?

[7 marks]

[TOTAL 20 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

PARTICLE PHYSICS With Advanced Study

For Fourth-Year Physics Students

Thursday 5th June 2003: 14.00 to 17.00

Answer THREE questions from Section A and ONE question from Section B.

Each question from Section A carries half as many marks as those carried by questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

In the following, units of $\hbar = c = \epsilon_0 = 1$ are used, so that

- 1 metre $\equiv 5.068 \times 10^{15}\,\mathrm{GeV}^{-1}$
- 1 second $\equiv 1.519 \times 10^{24}\,\mathrm{GeV}^{-1}$
- 1 Joule $\equiv 6.242 \times 10^{9}\,\mathrm{GeV}$
- 1 kilogram $\equiv 5.610 \times 10^{26}\,\mathrm{GeV}$

SECTION A

1. The Cabibbo-Kobayashi-Maskawa (CKM) unitary matrix gives the flavour-dependent relative couplings for the charged-current weak interactions of quarks, where V_{ij} is the factor for interactions involving quarks i and j. The numerical values of the magnitudes of the matrix elements can be taken to be

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.975 & 0.223 & 0.003 \\ 0.222 & 0.974 & 0.040 \\ 0.009 & 0.039 & 0.999 \end{pmatrix}.$$

(i) The tau lepton has a mass of 1.78 GeV and a lifetime of 0.29×10^{-12} s. It can decay semi-hadronically: to either a tau neutrino and one or more pions; or to a tau neutrino, a kaon and zero or more pions. Examples of these two types of decay are $\tau^- \to \nu_\tau \pi^-$ and $\tau^- \to \nu_\tau K^-$, respectively. Draw a quark-level Feynman diagram for each of these particular example decays. Ignoring mass effects, estimate their relative rates. [4 marks]

(ii) The only other decays of the tau are leptonic, to either $\tau^- \to \nu_\tau e^- \overline{\nu}_e$ or $\tau^- \to \nu_\tau \mu^- \overline{\nu}_\mu$. Ignoring mass effects, estimate the branching fractions for both of these decays and also for the two types of semi-hadronic decays described in part (i). [5 marks]

(iii) The charm meson D^+ (quark content $c\overline{d}$) has a measured lifetime of 1.05×10^{-12} s. Draw a quark-level Feynman diagram for the most common hadronic decay of this meson. [3 marks]

(iv) The total width of the tau is proportional to m_τ^5. Using this, and assuming asymptotic freedom holds for the D^+ decay ("spectator model"), estimate the D^+ lifetime and compare with the above value. The mass of the charm quark can be taken to be 1.4 GeV. State clearly any assumptions you make. [4 marks]

(v) The meson B^+ (quark content $u\overline{b}$) has a measured lifetime of 1.65×10^{-12} s. The mass of the bottom quark can be taken to be 4.2 GeV. Explain why the lifetime of the B^+ is comparable to that of the D^+, even though the b quark has a much larger mass than the c quark. [4 marks]

[TOTAL 20 marks]

2. The charged pion, with spin zero and mass 139.6 MeV, can decay to an electron, mass 0.511 MeV, or a muon, mass 105.7 MeV, through the decay

$$\pi^- \to l^- + \overline{\nu}_l,$$

where l stands for e or μ. These decays have branching fractions of 1.23×10^{-4} and 0.99988 respectively. Any neutrino mass should be neglected in the following.

(i) Draw a Feynman diagram for this decay. [3 marks]

(ii) Fermi's Golden Rule gives the partial width Γ_i for a particle of mass m to decay to a mode i to be

$$\Gamma_i = \frac{|M_i|^2 \rho_i}{2m},$$

where M_i is the matrix element and ρ_i the Lorentz invariant phase space. Briefly explain the physical significance of the terms in this equation. [3 marks]

(iii) The phase space available for the above pion decays is

$$\rho_l = \frac{1}{8\pi} \frac{m_\pi^2 - m_l^2}{m_\pi^2}.$$

Evaluate the ratio of the magnitudes of the phase space factors for these two decays and comment on your result. [2 marks]

(iv) The first-order matrix element for these decays is

$$|M_l|^2 = 2G_F^2 f_\pi^2 m_l^2 (m_\pi^2 - m_l^2).$$

where G_F is the Fermi constant and f_π is the pion form factor. Calculate an expression for the ratio of the partial widths for these two decays. Evaluate this ratio and comment on your result.

[3 marks]

(v) A left-handed state of a fermion has components of both helicity $\pm 1/2$ states, with amplitudes of $\sqrt{(1 \mp \beta)/2}$ respectively, where β is the velocity of the particle. Assuming the weak interactions couple only to left-handed particles (and hence right-handed antiparticles), draw a diagram showing the lepton and antineutrino helicities in these decays. [4 marks]

(vi) From the expression for the matrix element in part (iv) above, the decay rate becomes zero as the lepton mass goes to zero. Explain this observation in terms of your diagram from part (v) and hence qualitatively explain why the electron decay is heavily suppressed compared with the muon decay. [3 marks]

(vii) The charged kaon has a mass of 493.7 MeV. The branching fraction for the equivalent muon decay $K^- \to \mu^- \overline{\nu}_\mu$ is 0.6351. Briefly explain why this is lower than for the pion case and estimate the branching fraction for the decay $K^- \to e^- \overline{\nu}_e$.

[2 marks]

[TOTAL 20 marks]

3. The lowest mass baryons which contain only u and d quarks (which have charges of $+2/3$e and $-1/3$e, respectively) are the spin-1/2 nucleon family (the proton p and the neutron n) and the spin-3/2 Δ family (Δ^{++}, Δ^{+}, Δ^{0} and Δ^{-}).

(i) Write down the quark compositions of these six baryons within the quark model. [3 marks]

(ii) The masses of the baryons are; proton 938 MeV, neutron 940 MeV, Δ^{++} 1231 MeV, Δ^{+} 1232 MeV, Δ^{0} 1234 MeV and Δ^{-} 1235 MeV. What is the significance of the approximate equality of the masses of the nucleon states and of the Δ states? [3 marks]

(iii) The two major contributions to the mass of all these baryons can be assumed to be due to the masses of the quarks and the QCD binding energy. The latter is dominated by a spin-dependent term which results in a mass shift to the baryon given in terms of the spin-spin expectation value by

$$\delta m_{ij} = \frac{K}{m_i m_j} \langle \boldsymbol{s}_i . \boldsymbol{s}_j \rangle ,$$

for every pair of quarks with masses m_i and m_j and spins $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$ in the baryon. The quantity K is a constant. Obtain an expression for the total spin of the baryon squared, $\boldsymbol{S}^2$, in terms of the quark spins, assuming there is no orbital angular momentum in these states. [3 marks]

(iv) Neglecting any difference in the masses of the u and d quarks, so $m_u = m_d = m_q$, show that the total shift in mass due to the above term is given by

$$\delta m = \frac{K}{2m_q^2} \left[S(S+1) - \frac{9}{4} \right] .$$

[4 marks]

(v) Neglecting the small differences in the masses within the families, take the "average" for the nucleon mass as 939 MeV and for the Δ mass as 1233 MeV. Hence find values for m_q and K. [4 marks]

(vi) The small differences in mass within these families are due to the difference between the u and d quark masses and to electromagnetic forces. By considering the Δ^{++} and Δ^{-}, deduce which quark has the higher mass, stating clearly your reasoning. Hence, give an upper limit for the difference of their masses and comment on the validity of the approximation made in part (iv) that the quark masses can be taken to be the same. [3 marks]

[TOTAL 20 marks]

4. (i) Nöther's theorem states that for every symmetry of the Lagrangian density under the change of a continuous variable Λ, there is a conserved current J^μ given by

$$J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \frac{\partial q_i}{\partial \Lambda},$$

where q_i are the variables of the Lagrangian. Write down the equation which any conservation current such as J^μ satisfies in general and use this to show explicitly that $\int J^0 \, d^3r$ is a constant, where the integral is over all space. [4 marks]

(ii) The Lagrangian density for free electrons can be written as

$$\mathcal{L} = i\overline{\psi}\gamma^\mu(\partial_\mu\psi) - m\overline{\psi}\psi,$$

where $\overline{\psi} = \psi^\dagger\gamma^0$ and the γ matrices satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$. Show that there is a symmetry for the electron Lagrangian under the global phase transformation

$$\psi \to \psi e^{-ie\Lambda}.$$

Hence find the conserved current associated with this transformation. What is the physical quantity which is constant in this case? [6 marks]

(iii) Show that the Lagrangian is not invariant if a local phase change is applied, where $\Lambda(x^\mu)$, and find the extra term which arises as a result of this transformation. [4 marks]

(iv) Explain how the addition of an interaction term to the Lagrangian density given by

$$-eA_\mu\overline{\psi}\gamma^\mu\psi,$$

where A_μ is the electromagnetic potential, allows the total Lagrangian to become invariant to local phase changes. What property of A_μ is required for this to happen? Explain the significance of this result. [6 marks]

[TOTAL 20 marks]

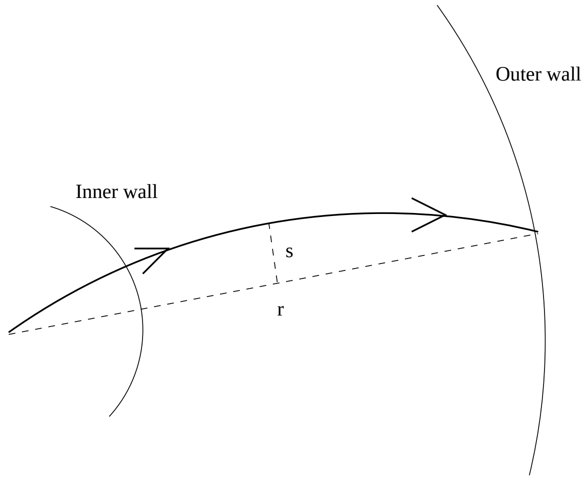
5. The ionisation of gas is used in wire chambers in many experiments to detect the passage of charged particles.

(i) Describe the principles of operation of a wire drift chamber, including the underlying physical processes.
[4 marks]

(ii) A particle trajectory is reconstructed by combining the measurements from many wires. A common geometry used has a cylindrical drift chamber with wires strung parallel to its axis and a uniform magnetic field along the axis. A charged particle emerging from the centre of the cylinder then bends perpendicularly to the field and a measurement of the radius of curvature R allows the momentum transverse to the field to be deduced, using the relation

$$p_T = 0.2998RB,$$

where B is in T, R in m and p_T in GeV. The effective measured quantity is the deviation of the trajectory from a straight line, the sagitta s, as shown in the diagram below. For a drift



chamber of outer radius r, show that the radius of curvature is given by

$$R = \frac{r^2 + 4s^2}{8s}.$$

[5 marks]

(iii) For particles with a large transverse momentum, the sagitta is very much smaller than the radius of the drift chamber and it has an error σ_s independent of its value. In this limit, show that the error on the transverse momentum σ_{p_T} is given by

$$\sigma_{p_T} = \left(\frac{8\sigma_s}{0.2998Br^2}\right) p_T^2.$$

[5 marks]

(iv) The BaBar experiment has a drift chamber with an outer radius of 81 cm in a 1.5 T magnetic field which can measure the sagitta to an accuracy of 70 μm. Calculate the expected fractional error on a 5 GeV transverse momentum measurement.
[4 marks]

(v) What other physical processes might cause the resolution on the momentum actually achieved to be worse than the above calculation would indicate?
[2 marks]

[TOTAL 20 marks]

6. Write short notes on THREE of the following topics.

(i) The structure and meaning of a Feynman diagram.

(ii) The role of deep inelastic scattering in determining the structure of the proton.

(iii) The process of hadronisation following high energy interactions which produce quarks.

(iv) Z production and decay at the LEP collider.

(v) Grand unified theories and running coupling constants.

(vi) The motivation and principles underlying a Cherenkov ring-imaging detector.

[TOTAL 20 marks]

SECTION B

7. For the next five years, the Tevatron proton-antiproton collider at Fermilab will be the only place able to search for the Higgs particle(s).

In the following, you may take the mass of the W to be 80 GeV, the mass of the Z to be 91 GeV, the quark masses to be; u 1 MeV, d 2 MeV, s 200 MeV, c 1.4 GeV, b 4.2 GeV and t 175 GeV, the charged lepton masses to be; e 0.51 MeV, μ 106 MeV and τ 1.78 GeV and the neutrinos to be massless.

(i) How does the coupling of the Standard Model (SM) Higgs to fermions depend on the fermion mass?

[3 marks]

(ii) What is the dominant decay mode of the SM Higgs for a Higgs mass around 120 GeV? Neglecting any differences in phase space, estimate the branching fraction for the dominant mode. Qualitatively explain what would change for a SM Higgs of mass 170 GeV.

[5 marks]

(iii) Draw the two principal Feynman diagrams for the SM Higgs production mechanisms at the Tevatron and describe their important features.

[4 marks]

(iv) For a SM Higgs of mass around 120 GeV, explain which diagram is more important for the experimental search. What are the charactistics of the resulting events which help to distinguish them from background? How would your answer differ if the Higgs mass was around 170 GeV?

[5 marks]

(v) Qualitatively describe the Higgs sector in the Minimal Supersymmetric Standard Model (MSSM). State why the Tevatron would be expected to be successful in its Higgs search if this model is valid.

[3 marks]

(vi) Write an essay in answer to ONE of the following.

(a) Explain the significance of the Higgs mechanism in the SM and give the current experimental status of knowledge of the SM Higgs mass.

(b) Discuss the critical issues in accelerator and detector performance for the SM and MSSM Higgs searches at the Tevatron.

(c) Discuss the implications for the SM, the MSSM and future Higgs searches in the two cases:

(i) the Tevatron discovers a Higgs boson with a mass around 120 GeV,

(ii) the Tevatron excludes a Higgs boson with mass below 170 GeV.

[20 marks]

[TOTAL 40 marks]

8. The CMS experiment for the LHC is building a large electromagnetic calorimeter using lead tungstate crystals. One of the main physics motivations is to detect the decay of the Higgs particle to two photons.

(i) Draw Feynman diagrams for the two dominant interaction processes responsible for the early part of the longitudinal development of high energy electromagnetic showers. Using these processes draw a simplified example of a high-energy photon shower up to a depth of around 3 radiation lengths. [4 marks]

(ii) By extrapolating this simple model to deeper depths, find an expression for the number of particles as a function of the depth in radiation lengths in terms of the energy E of the incoming photon. Assuming that in each interaction the original particle will share its energy equally between the outgoing particles, find the energy per particle as a function of depth. Hence, find the depth of the shower maximum if the critical energy is E_c. [2 marks]

(iii) On average, five photoelectrons are observed from the crystals for every MeV of energy of the incoming particle. Hence, calculate the functional form of the contribution to the fractional error on the energy due to the statistical fluctuations in the number of photoelectrons. [5 marks]

(iv) Show that the effective mass of two photons, $M_{\gamma\gamma}$, is given by

$$M_{\gamma\gamma}^2 = 2E_a E_b(1 - \cos\theta),$$

where E_a and E_b are the energies of the two photons and θ is the angle between the direction of flight of the photons. Assuming the angular resolution is accurate enough that its error can be neglected, show that the mass resolution is given by

$$\frac{\sigma_{M_{\gamma\gamma}}}{M_{\gamma\gamma}} = \frac{1}{2}\sqrt{\left(\frac{\sigma_{E_a}}{E_a}\right)^2 + \left(\frac{\sigma_{E_b}}{E_b}\right)^2},$$

where $\sigma_{M_{\gamma\gamma}}$ is the error on the two photon invariant mass, and σ_{E_a} and σ_{E_b} are the errors on the photon energies E_a and E_b, respectively. [5 marks]

(v) Consider a Higgs boson of a mass of 130 GeV decaying into two photons of equal energies. The calorimeter fractional energy resolution for photons in this energy range is 0.8%. Estimate the mass resolution for the reconstructed Higgs mass. At the LHC, the cross section for the production of such a Higgs boson and its subsequent decay into two photons is 50 fb. There is background from other processes which contributes random combinations of photons at a rate equivalent to 600 fb for each GeV in the range of reconstructed mass. The total efficiency for reconstructing a photon in the calorimeter is 60%. Estimate the numerical significance of a potential discovery of the Higgs for an integrated luminosity of 100 fb^{-1} if events within $\pm 1\sigma_{M_{\gamma\gamma}}$ of the Higgs mass are accepted.

[4 marks]

(vi) Write an essay in answer to ONE of the following.

(a) Discuss how the physics goals of an experiment influence how a system such as the CMS calorimeter is designed; in particular, the choice of crystal material, the geometrical arrangement of the crystals, the choice of photodetector, the mechanical structure and the dynamic range and resolution of the electronics.

(b) Describe the principles of operation of the CMS crystal calorimeter, including the fundamental physical interactions, the subsequent processes and how these are detected in an experiment. To which groups of particles is the calorimeter sensitive and what is the expected response?

(c) The energy resolution of an electromagnetic calorimeter is often expressed as

$$\frac{\sigma_E}{E} = \sqrt{\left(\frac{a}{\sqrt{E}}\right)^2 + \left(\frac{b}{E}\right)^2 + c^2}.$$

Explain the effects that are relevant for each of the three terms on the right hand side, usually known as stochastic, noise and constant terms, respectively, and describe how they can be reduced.

[20 marks]

[TOTAL 40 marks]

9. If neutrinos have mass and lepton number is not conserved then the neutrino flavour eigenstates, $(\nu_e, \nu_\mu, \nu_\tau)$, need not be the same as the mass eigenstates, (ν_1, ν_2, ν_3), and mixing can occur.

(i) Consider the case with mixing only between the electron and muon states, i.e.

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}.$$

A pure electron neutrino beam of momentum p is created at time $t = 0$. What is the composition of the neutrino state at a later time t? Show that the amplitude of muon neutrinos in the beam is given by

$$A_\mu = \cos\theta \sin\theta \left(e^{-iE_2 t} - e^{-iE_1 t}\right),$$

where E_i is the energy of the state ν_i. [3 marks]

(ii) Hence, show that the proportion of electron neutrinos in the beam after time t is

$$P_e = 1 - \sin^2 2\theta \sin^2 \left[\frac{(E_2 - E_1)t}{2}\right].$$

Assuming that $E_i \gg m_i$, then show that at a distance L from the source, this proportion is

$$P_e \approx 1 - \sin^2 2\theta \sin^2 \left[\frac{\Delta(m^2)L}{4E}\right],$$

where $\Delta(m^2)$ is the difference of the squares of the masses. [4 marks]

(iii) A reactor produces electron antineutrinos isotropically with an energy of a few MeV. You may assume that the formalism for antineutrinos is the same as for neutrinos. Two detectors, sensitive only to charged current neutrino interactions with protons and neutrons, are placed as follows:

- A near detector of 100 tons at 30 m from the reactor with an efficiency for recording the interactions of 50%.
- A far detector of 30 kilotons at 39 km from the reactor with an efficiency for recording the interactions of 70%.

Write down the charged current reactions for antineutrinos on protons and neutrons which require the least energy. Hence, explain why these detectors will only be sensitive to the electron antineutrino component of the beam. (Assume the neutrons and protons bound in the target nuclei have the same effective mass.) [2 marks]

(iv) In one year, 930 000 interactions are recorded in the near detector. Under the assumption of no oscillations, how many interactions would be expected to be recorded in the far detector in a year? If oscillations occur with $\Delta(m^2) = 4 \times 10^{-4}\,\text{eV}^2$ and $\sin^2 2\theta = 0.816$, and the antineutrinos are produced with an energy of 2 MeV, how would the expected number of interactions in the far detector change? [4 marks]

(v) In practice the neutrinos are emitted with a range of energies. Assuming a uniform spectrum from 1 to 3 MeV, argue that this energy spread will have minimal effect on the rate at the near detector but will change the proportion of electron antineutrinos at the far detector to

$$P_e \approx 1 - \frac{1}{2}\sin^2 2\theta .$$

Hence find the expected number of electron antineutrino interactions at the far detector in one year's running.

[5 marks]

(vi) Discuss the merits of a broad or narrow energy spread for the neutrinos in oscillation experiments.

[2 marks]

(vii) Write an essay in answer to ONE of the following.

(a) Summarise the experimental evidence for oscillations in neutrinos from the atmosphere.

(b) If neutrinos are able to undergo oscillations, discuss the consequences for the Standard Model and the implications for cosmology.

(c) The contribution of the Sudbury Neutrino Observatory (SNO) results to solar neutrino studies.

[20 marks]

[TOTAL 40 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

PLASMA PHYSICS With Advanced Study

For Fourth-Year Physics Students

Thursday 22nd May 2003: 14.00 to 17.00

Answer THREE questions from Section A and ONE question from Section B.

Each question from Section A carries half as many marks as those carried by questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) A test particle of charge, Q is placed in a plasma of electron density, n_e, which has an electron temperature, T_e and an ion temperature T_i. Show that the electromagnetic potential, ϕ, set up in the plasma at a distance r is approximately given by the expression (for $k_B T_e, k_B T_i \gg e\phi$);

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) = \frac{\phi}{\lambda_D^2}$$

Obtain an expression for λ_D and discuss its physical significance. [6 marks]

(ii) Use the substitution $u = r\phi$ to solve this equation and sketch this potential. [5 marks]

(iii) What is the total charge enclosed in a radius λ_D ? Sketch the ion and electron charge density (for $k_B T_e, k_B T_i \gg e\phi$). [9 marks]

[TOTAL 20 marks]

Note: in spherical co-ordinates (r, θ, ϕ)

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

2. (i) What is an adiabatic invariant in a dynamical system? Define the "first" and "second" adiabatic invariants in plasma physics? Under what conditions are they approximately constant?

[6 marks]

(ii) What is a magnetic mirror? How can it be used as a magnetic confinement device? [4 marks]

(iii) Using the first adiabatic invariant, μ show that the velocity, v_z of an electron in a magnetic field, B_z which varies in the z direction is given by;

$$v_z = \left(v_0^2 - \frac{2\mu B_z(z)}{m_e} \right)^{1/2}$$

where v_0 is a constant.

[3 marks]

(iv) If the magnetic field in a magnetic mirror is given by

$$B_z(z) = B_0 \frac{| z |}{l}$$

such that l is a constant, calculate the second adiabatic invariant. If l increases at a rate R, determine a condition on R such that the second adiabatic invariant is conserved.

[7 marks]

[TOTAL 20 marks]

3. (i) The Rutherford scattering formula for two charged particles is given by:

$$\cot\left(\frac{\chi}{2}\right) = \frac{4\pi\varepsilon_0 m v^2 b}{q_1 q_2}$$

Indicate χ, b, and v on a sketch.

[4 marks]

(ii) Show that the collision time (i.e., the time required to produce a root mean square deflection of 90°) for multiple small angle scattering of an electron from stationary ions in a plasma is:

$$\tau_e = \frac{\pi^3 \varepsilon_0^2 m_e^2 v_e^3}{2Z^2 e^4 n_i \ln\Lambda} .$$

[8 marks]

(iii) What is $\ln\Lambda$? How is it usually calculated? Show that the resistivity of a plasma is;

$$\eta \propto Z \ln\Lambda \, (k_B T_e)^{-3/2} .$$

Assume quasi-neutrality.

[5 marks]

(iv) Discuss qualitatively how the resistivity of a plasma is modified if a magnetic field is present.

[3 marks]

[TOTAL 20 marks]

4. (i) What is a tokamak? Why is a toroidal current necessary for plasma confinement in a tokamak? How do particle drifts affect the orbits of trapped particles in a tokamak magnetic field configuration? Show the shape of trapped and untrapped particle orbits in a tokamak (including the Larmor motion). [10 marks]

(ii) The total magnetic field in a tokamak can be approximated by;

$$B = B_0 \left(1 - \frac{r}{R} \cos\theta\right)$$

where r is the minor radius of the tokamak and R is the major radius and θ is the poloidal angle such that $\theta = 0$ is at the outside of the torus (where B is a minimum).

Show that the trapping condition is;

$$\frac{v_{\parallel 0}}{v_{\perp 0}} < \frac{\sqrt{2}}{a^{1/2}}$$

$$\text{for } a \gg 1$$

where $a = R/r$ = aspect ratio, $v_{\parallel 0}$ is the velocity parallel to the magnetic field at $\theta = 0$, and $v_{\perp 0}$ is the velocity perpendicular to the magnetic field at $\theta = 0$. [5 marks]

(iii) Considering only the curvature drift for the motion of an electron which is just barely trapped, estimate the width of a banana orbit in a tokamak in terms of a, $v_{\parallel}$, and B_p (the poloidal magnetic field). [5 marks]

[TOTAL 20 marks]

5. (i) What are flux surfaces? Show that in ideal MHD magnetic field lines lie on a flux surface in equilibrium.

[6 marks]

(ii) Show that in ideal MHD

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

where $\mathbf{u}$ is the fluid velocity.
How is this equation changed if resistivity is included?

[5 marks]

(iii) Show that magnetic field lines are "frozen" in the plasma in ideal MHD.
What occurs if resistivity is non-negligible?

[9 marks]

[TOTAL 20 marks]

Note that;

$\nabla(\mathbf{A}\cdot\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$

and;

$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla \times \nabla \times \mathbf{A}.$

6. Write short notes on THREE of the following topics (using equations and diagrams to illustrate).

(i) Ignition

(ii) Alfvén waves

(iii) Polarisation drift

(iv) Plasma parameter

(v) Bump-on-tail instability

(vi) Critical density

[TOTAL 20 marks]

SECTION B

7. Write an essay which explains why turbulence is a problem for magnetic confinement fusion experiments and which describes the nature of the turbulence.
[40 marks]

8. Write an essay discussing the physics and applications of ion thrusters.
[40 marks]

9. Write an essay about the physics of particle acceleration using intense lasers.
[40 marks]

End

UNIVERSITY OF LONDON

MSci EXAMINATION June 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY
With Advanced Study

For Fourth-Year Physics Students

Monday 2nd June 2003: 14.00 to 17.00

Answer THREE questions from Section A and ONE question from Section B.

Each question from Section A carries half as many marks as those carried by questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION. Enter the number of each question attempted in

the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. The Hamiltonian of the Dirac field can be written as

$$H = \int d^3x \left(-i\bar{\psi}\gamma^k \partial_k \psi + m\bar{\psi}\psi\right)$$

where $k = 1, 2, 3$.

(i) Write down Dirac's equation. [1 mark]

(ii) By applying a suitable operator to the Dirac equation (thus "squaring" it) derive the algebra of constraints satisfied by the gamma matrices. [4 marks]

(iii) Write down Heisenberg's equation for a general quantum system in the Heisenberg picture. [2 marks]

(iv) Specialize the previous equation to the case of a quantum Dirac field. [2 marks]

(v) Prove the identity

$$[AB, C] = A\{B, C\} - \{A, C\}B \tag{1.1}$$

[1 mark]

(vi) Using Eqn. (1.1) and the equal-time anticommutation relations:

$$\{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = \delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{ab} \tag{1.2}$$

show that the equation you wrote in part (iv) is equivalent to Dirac's equation. [10 marks]

[TOTAL 20 marks]

2. The Klein-Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

can be seen as the simplest relativistic generalization of the Schrödinger's equation.

(i) Based on the non-relativistic relation between energy and momentum provide a heuristic argument justifying Schrödinger's equation. Show that the recipe for finding the energy and momentum operators is already covariant.
[3 marks]

(ii) Write the relativistic generalization of the relation considered in the previous question and show how it leads to the Klein-Gordon equation.
[4 marks]

(iii) Find the eigenfunctions of the operators representing energy and momentum, used in your previous argument. Derive the condition under which they solve the Klein-Gordon equation.
[4 marks]

(iv) Write the relativistic probability current j^μ. Show that the probability density ρ is not positive definite, giving one concrete example where $\rho < 0$.
[4 marks]

(v) Even more "anti-social" than negative probabilities, are particles that travel faster than light. These so-called tachyons have imaginary masses. Write the tachyonic free solution to the Klein-Gordon's equation in the "rest frame". How does it differ from well-behaved solutions? Is such a rest frame physical?
[5 marks]

[TOTAL 20 marks]

3. Consider a real scalar field $\phi(x)$.

(i) Write down an expansion of the field in Fourier modes. [1 mark]

(ii) Identify the Lorentz invariant volume element and identify, from the Fourier amplitudes, the creation and annihilation operators. [3 marks]

(iii) Canonical quantization amounts to postulating

$$[\phi(\mathbf{x}, t), \dot{\phi}(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

with all other commutators either following trivially or being zero. From these commutators derive the commutation relations for the creation and annihilation operators. [10 marks]

(iv) How can you set up the Fock space of this system? [3 marks]

(v) Show that the quantum particles associated with ϕ are bosons. [3 marks]

[TOTAL 20 marks]

4. The *unequal* time commutation relations for a real scalar field follow from the equal-time commutators, with the result:

$$[\phi(x), \phi(y)] = i\Delta(x-y)$$

where Δ is a function to be determined and x and y are 4-vectors.

(i) Prove that

$$\Delta(x) = -i\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0}\left[e^{-ik\cdot x} - e^{ik\cdot x}\right]$$

and so

$$\Delta(x) = -i\int \frac{d^4k}{(2\pi)^4}\epsilon\left(k^0\right) 2\pi\delta\left(k^2 - m^2\right) e^{-ik\cdot x}$$

with $\epsilon(x) = x/|x|$, and k also a 4-vector. [5 marks]

(ii) Prove that $\Delta(x) = 0$ for $x^0 = 0$, thus recovering the *equal* time commutators for ϕ. Using Lorentz invariance show that $\Delta(x) = 0$ for $x^2 < 0$. [5 marks]

(iii) Show that for $x^0 = 0$ we have

$$\frac{\partial}{\partial x^0}\Delta(x) = -\delta^{(3)}(\mathbf{x})$$

and so recover the *equal* time commutators postulated in lectures. [5 marks]

(iv) Express Δ in compact form as a contour integral in a complex time variable, identifying both the integrand and the contour.

[5 marks]

[TOTAL 20 marks]

5. The gauge principle is a powerful method for introducing interactions. In this question we shall introduce electromagnetism by enforcing a $U(1)$ (phase) gauge symmetry.

(i) Write down the lagrangian for a complex scalar field. [4 marks]

(ii) Show that the theory is invariant under a global shift in phase. [4 marks]

(iii) Now allow for space-time dependence in your phase transformation. Considering an infinitesimal phase transformation show how the invariance of the theory is now broken. [4 marks]

(iv) How should you modify your lagrangian in order to preserve invariance? (Hint: Introduce the gauge field and show how it may be used to define a "covariant derivative"). [4 marks]

(v) Write down the most general gauge invariant lagrangian. [4 marks]

[TOTAL 20 marks]

6. The electrostatic potential ϕ and the vector potential $\mathbf{A}$ may be joined together in a 4-vector $A^\mu = (\phi, \mathbf{A})$.

(i) Write down the Lorentz transformations associated with this 4-vector, and using them compute ϕ and $\mathbf{A}$ associated with a charge moving at relativistic speeds. [4 marks]

(ii) The Lorentz gauge is defined by

$$\dot{\phi} + \mathrm{div}\mathbf{A} = 0$$

and in this gauge Maxwell equations may be written as

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0$$

and likewise for $\mathbf{A}$. Re-express these equations in covariant notation. [4 marks]

(iii) If we define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

how can we express the various entries of $F_{\mu\nu}$ in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. [4 marks]

(iv) Using the results in part (iii) find the Lorentz transformations associated with $\mathbf{E}$ and $\mathbf{B}$. [8 marks]

[TOTAL 20 marks]

SECTION B

7. Write an essay explaining how the Feynman rules for scalar fields give rise to divergent loop diagrams. Explain in particular the divergence of the integral given by a two-point diagram in a phi-cubed scalar field theory that is built out of two three-point vertices connected by propagators so as to make a single loop. Describe in particular how one may use a Wick rotation from Minkowskian to Euclidean signature momentum space and characterize the leading divergence of this one-loop diagram when analyzed in momentum space. Explain why it was realised in the late 1940's that such diagrams had to be taken seriously, despite their infinite divergences.

[40 marks]

8. Pick one (any) type of a topological defect and discuss how and when it could form and what would be its cosmological implications.

[40 marks]

9. In the quantum theory of the free scalar field, the Hamiltonian is given by

$$H = \frac{1}{2}\int d^3x \left(\hat{\pi}^2 + \left(\nabla\hat{\phi}\right)^2 + m^2\hat{\phi}^2\right)$$

where the scalar field operator $\hat{\phi}(\mathbf{x})$ and its conjugate $\hat{\pi}(\mathbf{x})$ obey the usual equal time canonical commutation relations

$$\left[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] = i\delta(\mathbf{x}-\mathbf{y})$$

and

$$\left[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})\right] = 0 = \left[\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right].$$

(i) In the functional Schrödinger quantization of scalar field theory, the states are represented by wave functionals $\Psi\left[\phi(\mathbf{x})\right]$ of the field configurations. Explain how the canonical commutation relations are represented and write down the functional Schrödinger equation for the time-development of the state. [10 marks]

(ii) The annihilation operator $a_{\mathbf{k}}$ is given by

$$a_{\mathbf{k}} = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\omega_{\mathbf{k}}\hat{\phi}(\mathbf{x}) + i\hat{\pi}(\mathbf{x})\right)$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. Write down the functional Schrödinger picture version of the equation defining the ground state,

$$a_{\mathbf{k}}\Psi_0\left[\phi(\mathbf{x})\right] = 0\,.$$

Show that it may be solved using the ansatz,

$$\Psi_0\left[\phi(\mathbf{x})\right] = \exp\left(-\frac{1}{2}\int d^3y d^3z\,\phi(\mathbf{y})E(\mathbf{y},\mathbf{z})\phi(\mathbf{z})\right)$$

as long as $E(\mathbf{y},\mathbf{z}) = E(\mathbf{z},\mathbf{y})$ obeys a simple integral equation, which you should derive. Verify that your integral equation for $E(\mathbf{y},\mathbf{z})$ is satisfied by

$$E(\mathbf{y},\mathbf{z}) = \frac{1}{(2\pi)^3}\int d^3k'\omega_{\mathbf{k}'}\,e^{-i\,\mathbf{k}'\cdot(\mathbf{y}-\mathbf{z})}\,.$$

[18 marks]

(iii) Answer *either* part (a) *or* part (b).

(a) In the functional Schrödinger quantization of electromagnetism, the state are represented by wave functionals $\Psi\left[A_i(\mathbf{x})\right]$ which depend on the 3-vector potential $A_i(\mathbf{x})$. How is the momentum $\pi^i(\mathbf{x})$ conjugate to $A_i(\mathbf{x})$ represented and what is its commutation relation with $A_i(\mathbf{x})$? Explain what happens to these wave functions under an infinitesimal gauge transformation, $A_i(\mathbf{x}) \to A_i(\mathbf{x}) + \partial_i\Lambda(\mathbf{x})$, where $\Lambda(\mathbf{x})$ is a small arbitary function.

(b) The functional Schrödinger picture may be generalized to curved space-times with metric

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2$$

where $a(t)$ is an arbitrary function of time. Give a brief qualitative account of some of the new features and problems that arise in this situation. (Note: A variety of different answers may gain full marks for this part. It is not necessary to give any detailed calculations.)

[12 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

QUANTUM OPTICS With Advanced Study

For Third- and Fourth-Year Physics Students

Tuesday 27th May 2003: 14.00 to 17.00

Answer THREE questions from Section A and ONE question from Section B.

Each question from Section A carries half as many marks as those carried by questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) What are the main features of the Black body spectrum? [3 marks]

(ii) The probability of there being n photons in the mode with frequency ω inside a black body at temperature T is given by the Boltzmann formula. The density of states of a photon gas is given by

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3}$$

Show that the Planck black body formula is given by the following expression

$$W_T = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}$$

Identify all the terms in the above formula. [7 marks]

(iii) (a) Calculate the standard deviation in the photon number in terms of the average number of photons $\langle n \rangle$. [6 marks]

(b) Which term in the expression of the standard deviation carries the signature of particle behaviour and which reflects wave behaviour and why (a qualitative answer is sufficient here)? [4 marks]

You may use the following expression:

$$\frac{\sum_{n=0}^{\infty} n e^{-n\hbar\omega/kT}}{\sum_{m=0}^{\infty} e^{-m\hbar\omega/kT}} = \frac{1}{e^{\hbar\omega/kT} - 1}$$

[TOTAL 20 marks]

2. Describe briefly the main features of laser light.

[3 marks]

(i) Imagine a one dimensional vacuum cavity with perfectly reflecting mirrors at $x = 0$ and $x = L$. Write down the expression for the wavelengths and frequencies of the allowed modes of the cavity as a function of L?

[4 marks]

(ii) Suppose that each mode i can be represented as a wave of the form

$$E_i = E_0 e^{i\omega_i t}$$

where the amplitude E_0 is the same for all modes, and ω_i is the frequency of the i-th mode. Write down the expression for the total intensity of light inside the cavity taking into account N modes? At what times do the intensity peaks occur and what is the width of each pulse?

[5 marks]

(iii) A more realistic form of the ith mode amplitude is

$$E_i = E_0 e^{i(\omega_i t + \delta_i)}$$

where δ_i is the extra phase of the ith mode. Comment on the physical origin of this extra phase. What is the total intensity as the number of modes becomes very large assuming that the extra phases, δ_i, are completely randomly distributed?

[5 marks]

(iv) How would you make very short laser pulses with a very high intensity?

[3 marks]

[TOTAL 20 marks]

3. (i) Explain briefly the semi-classical approximation in the treatment of light-matter interactions.

[3 marks]

(ii) A nuclear spin has two possible states in an external magnetic field, up $|\uparrow\rangle$ (i.e. aligned with the field) and down $|\downarrow\rangle$ (i.e. anti-aligned with the field). Suppose that the nucleus is in an external (static) magnetic field of strength B, which points in the z direction.

(a) Write down the Hamiltonian for the nucleus using the Pauli matrix notation and identify its eigenvalues.

[4 marks]

(b) Suppose that the initial state of the system is aligned with the field in the z direction, $|\uparrow\rangle$. Suppose then that the field is instantaneously switched to the x direction. Solve the Schrödinger equation to obtain the exact evolution of the nuclear spin in terms of the eigenstate of the Pauli spin matrix σ_x. What is the phase difference between the two orthogonal spin eigenstates of σ_x as a function of time?

[6 marks]

(c) After what time will the spin switch to *its* orthogonal state $|\downarrow\rangle$?

[4 marks]

(iii) What is the relationship between the energy associated with the spin and the time it takes to evolve between orthogonal states? Comment on the validity of the time-energy uncertainty relation to estimate the time of this transition.

[3 marks]

The Pauli matrices are given in the $|\uparrow\rangle, |\downarrow\rangle$ basis by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[TOTAL 20 marks]

4. A quantum particle moving non-relativistically in one dimension has mass m and potential energy $\frac{1}{2}m\omega^2x^2$. Write down its Hamiltonian H. Express H in terms of the operators

$$\begin{aligned} a &= \frac{\beta}{\sqrt{2}}(x + i\frac{p}{m\omega}) \\ a^\dagger &= \frac{\beta}{\sqrt{2}}(x - i\frac{p}{m\omega}) \end{aligned}$$

where $\beta^2 = m\omega/\hbar$. You may assume that $[x, p] = i\hbar$.

[4 marks]

(i) Evaluate the commutators $[a, a^\dagger]$, $[H, a^\dagger]$ and $[H, a]$.

[4 marks]

(ii) Hence determine the allowed energy levels of the particle, explaining carefully the logic that you use. What do these levels represent when we apply them to a single mode of the quantized electro-magnetic field?

[7 marks]

(iii) Let $|0\rangle$ denote the ground state. Show that

$$\begin{aligned} \langle 0|(a + a^\dagger)|0\rangle &= 0 \\ \langle 0|(a + a^\dagger)^2|0\rangle &= 1 \end{aligned}$$

What do these relationships signify in relation to the quantized electro-magnetic field?

[5 marks]

[TOTAL 20 marks]

5. An atom has two energy levels, i and j, separated in energy by $\hbar\omega_{ij}$. It is subject to a small external time dependent monochromatic electro-magnetic perturbation for a time T, oscillating at the frequency ω. You may assume that the perturbation has matrix elements $V_{ji} = V_{ij}^*$ between these states. Show that if the atom is initially in the state i, the probability of a transition to the state j is approximately

$$P_{ij} = 4|V_{ij}|^2 \frac{\sin^2((\omega_{ij} - \omega)T/2)}{(\hbar(\omega_{ij} - \omega))^2}$$

[10 marks]

(i) Argue that the probability of the transition back from j to i is the same as that for the transition from i to j.

[3 marks]

(ii) Show that, within the formalism employed, the transition rate grows linearly with time.

[3 marks]

(iii) Why is the Einstein B coefficient independent of time (only a qualitative explanation required)?

[4 marks]

[TOTAL 20 marks]

6. Discuss briefly THREE of the following:

(i) Stimulated emission.

(ii) The zero-point energy.

(iii) The classical model of the light-matter interaction.

(iv) Rabi Oscillations.

(v) Phase Matching.

(vi) Coherent States.

[TOTAL 20 marks]

SECTION B

7. Choose ONE of the following: your essay should outline why the paper is regarded as important, what it sets out to demonstrate, how the authors' experiments worked, and how convincing were the results. You should NOT just copy out the paper, but be critical about it. A good essay should contain sketches and relevant diagrams and should be self-contained.

(i) Write an essay discussing the significance and content of the following paper:

"Observation of Bose-Einstein Condensate in a Dilute Atomic Vapour", M H Anderson, J R Enscher, M R Matthews, C E Wieman and E A Cornell, Science 269, 198 (1995).

(ii) Write an essay discussing the significance and content of the following paper:

"Quantum Rabi Oscillations: A Direct Test of Field Quantization in a Cavity", M Brune, F Schmidt-Kaler, A Maali, J Dreyer, F Hagley, J M Raimond and S Haroche, Phys Rev Lett 76, 1800 (1996).

(iii) Write an essay discussing the significance and content of the following paper:

"Experimental Quantum Teleportation", D Bouwmeester, J W Pan, K Mattle, M Eibl, H Wienfurter and A Zeilinger, Nature 390, 576 (1997).

[TOTAL 40 marks]

End

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2003

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

SPACE PHYSICS With Advanced Study

For Fourth-Year Physics Students

Wednesday 21st May 2003: 14.00 to 16.00

Answer THREE questions from Section A and TWO questions from Section B.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

SECTION A

1. (i) An important equation in space plasma physics is Ohm's law:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.1}$$

where $\mathbf{j}$ is the current density vector, σ the electrical conductivity, $\mathbf{v}$ the plasma velocity and $\mathbf{E}$, $\mathbf{B}$ the electric and magnetic fields respectively. Using Ohm's law, describe a common assumption which can be made in space plasmas by which one is able to show that the electric field arises purely from plasma motion.
[2 marks]

(ii) Write down the magnetic induction equation which describes the evolution of the magnetic field with time. Describe each of the terms in this equation, as well as what happens in the limit of high conductivity.
[4 marks]

(iii) Using your answers from (i) and (ii), describe under what conditions the magnetohydrodynamic (MHD) approximation holds. What is the consequence of this approximation and give two examples of where it breaks down.
[3 marks]

(iv) Use Ampere's law to show that the $\mathbf{j} \times \mathbf{B}$ force can be written as:

$$\mathbf{j} \times \mathbf{B} = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla\mathbf{B}) \tag{1.2}$$

where μ_0 is the plasma permeability. Describe the nature of each term on the right hand side of the equation (1.2) and their effect on the plasma.
[5 marks]

Make use of the following vector identity as necessary, where $\mathbf{G}$ and $\mathbf{F}$ are arbitrary vectors.

$$(\mathbf{G}\cdot\nabla)\mathbf{F} = \frac{\mathbf{1}}{\mathbf{2}}[\nabla \times (\mathbf{F} \times \mathbf{G}) + \nabla(\mathbf{F}\cdot\mathbf{G}) - \mathbf{F}(\nabla\cdot\mathbf{G}) + \mathbf{G}(\nabla\cdot\mathbf{F}) - \mathbf{F} \times (\nabla \times \mathbf{G}) - \mathbf{G} \times (\nabla \times \mathbf{F})]$$

(v) Describe the concept of the Debye length, λ_D, of a plasma and quasi-neutrality. Show that from:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) = \frac{-en_o}{\varepsilon_o}\left(1 - \exp\left(\frac{e\Phi(r)}{kT}\right)\right)$$

that for sufficiently high temperatures this can be written as:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) \cong \lambda_D\Phi(r)$$

where e is the electrical charge, n_o the number density, ε_o the permittivity, k is Boltzmann's constant and T the temperature. Derive the form of λ_D .
[4 marks]

(vi) Using the form of λ_D derived in (v), a numerical expression for the Debye length can be derived which is given by:

$$\lambda_D = 69\sqrt{\frac{T}{n_o}}$$

where the units of λ_D in this case are metres, T is in units of 10^6K, n_o is in cm^{-3}. Using typical solar wind parameters near 1 astronomical unit of $T \sim 10^5$K and $n_o \sim 10$cm^{-3}, calculate λ_D and decide whether the solar wind plasma in this case is quasi-neutral.
[2 marks]

[TOTAL 20 marks]

2. (i) The radiation belts of the Earth consist of energetic charged particles trapped by the Earth's dipole magnetic field. Such electrically charged particles undergo three different types of motion. List the three types and briefly describe the resulting motion for each. [3 marks]

(ii) The equation of motion of a particle of charge q, mass m and velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, is:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ .$$

Let us assume that $\mathbf{E} = \mathbf{0}$ and that there is a constant homogenous magnetic field $\mathbf{B}$ in the z-direction only, so $\mathbf{B} = (0, 0, \mathrm{B})$. Describe briefly without derivation the resultant particle motion in the (x,y) plane, as well as in three dimensions. [2 marks]

(iii) In the more general case of a constant homogenous $\mathbf{E}$ and $\mathbf{B}$ field, use the equation of motion to show that the velocity of the particle perpendicular to $\mathbf{B}$ can be written as:

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \left(\frac{m}{qB}\right)\frac{d}{dt}\left(\mathbf{v} \times \hat{\mathbf{b}}\right)$$

where $\hat{\mathbf{b}}$ is the unit vector along $\mathbf{B}$. Describe each of the terms on the right hand side of this equation. [6 marks]

(iv) A gradient $\mathbf{B}$ drift velocity arises for charged particles moving in a magnetic field which varies in strength in a direction perpendicular to $\mathbf{B}$, given by:

$$\mathbf{v}_{gd} = \frac{mv_\perp^2}{2qB^3}(\mathbf{B} \times \nabla B)$$

where B=$|\mathbf{B}|$, is the magnetic field magnitude and $\mathbf{v}_\perp$ is the velocity of the particle perpendicular to the magnetic field. We concentrate our attention in the Earth's equatorial plane and use a cylindrical co-ordinate system ($\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$). Then the magnetic field at a distance r from the centre of the Earth is given by:

$$\mathbf{B}_{eq}(r) = B_o\left(\frac{R_E}{r}\right)^3\hat{\mathbf{z}}$$

where B_o is the field strength at the surface of the Earth at the equator, R_E is the radius of the Earth, and $\hat{\mathbf{z}}$ is the unit vector pointing North. Derive the gradient drift motion which the particles will undergo. Use the following as necessary:

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \ .$$

[3 marks]

(v) Using your answer from (iv) explain the behaviour of the ions and electrons. Describe the current and $\mathbf{B}$ field which result from their motion. Use a simple schematic (looking down on the Earth's equatorial plane) to show the effects of this drift motion. [6 marks]

[TOTAL 20 marks]

3. The magnetic field of the Earth is represented by a dipole field, with the rotation and magnetic axes aligned. In spherical polar co-ordinates the magnetic field components of such a dipole field can be written as:

$$B_r = -\frac{2M\cos\theta}{r^3}, \;\; B_\theta = -\frac{M\sin\theta}{r^3}, \;\; B_\phi = 0$$

where M is the dipole moment of the Earth, r is the radial distance from the center of the planet, and θ is the co-latitude.

(i) Show that the magnitude of the **B** field is given by:

$$B(r,\theta) = B_{eq}\frac{R_E^3}{r^3}\sqrt{1+3\cos^2\theta}$$

where R_E is the radius of the Earth. What is B_{eq}? [2 marks]

(ii) Write down the general vector equation describing a field line, as well as explicitly writing down the resultant field line equations in spherical polar co-ordinates. Use these to show that the equation of a field line in the Earth's dipole magnetic field can be written as:

$$r = r_{eq}\sin^2\theta\,.$$

Define r_{eq}. [6 marks]

(iii) A magnetospheric cavity forms as a result of the interaction between the solar wind plasma and the Earth's magnetic field. Explain briefly and conceptually how and which two basic plasma physics principles can be used to explain the formation of the magnetosphere. [3 marks]

(iv) Under what conditions can magnetic reconnection occur, and give two examples found in the Earth's magnetosphere where reconnection regularly occurs. Use a set of simple schematics to show the reconnection of southward interplanetary magnetic field lines across the magnetopause boundary with planetary field lines, as well as the behaviour of the resulting reconnected field lines. [6 marks]

(v) Describe briefly the conditions necessary for the generation of a magnetospheric substorm, as well as the ordered sequence of events which occurs during such a substorm. [3 marks]

[TOTAL 20 marks]

4. The solar wind is a continuous stream of plasma which flows radially away from the Sun at a velocity v. The differential equation which describes this outflow can be derived from the equations of conservation of mass and momentum and is given by:

$$\left(v^2 - \frac{2k_BT}{m}\right)\frac{1}{v}\frac{dv}{dr} = \frac{4k_BT}{mr} - \frac{GM_S}{r^2} \tag{4.1}$$

where it has been assumed that the plasma pressure $p = 2nk_BT$, in which k_B is Boltzmann's constant, T the temperature, G the gravitational constant, m the mass of the solar wind particles and M_S the solar mass.

(i) Sketch the behaviour of the unique solution of equation (4.1) which describes the behaviour of the outflow of the solar wind. Show the critical radius and the corresponding velocity (derive values for each). Explain why this unique solution is the solar wind solution. Show also on the sketch the four families of solutions which are possible and briefly discuss the physical admissibility of each.
[10 marks]

(ii) The solar atmosphere rotates about an axis nearly perpendicular to the ecliptic plane with a rotation period of 25.4 days at the equator. Define the angular frequency (ω_S) of the Sun in the equatorial plane. This solar rotation results in the interplanetary magnetic field (IMF) lines taking the shape of a spiral as they move outwards away from the Sun. Draw a simple schematic, in the equatorial plane, which shows the behaviour of a single field line at successive time intervals beginning at an initial solar longitude of ϕ_o (at $t = t_o$) and ending at ϕ at time t.
[2 marks]

(iii) On transforming into a frame of reference rotating with the Sun (so the source of the plasma and field lines remains fixed), derive the Archimedian spiral equation:

$$\phi(r) = \phi_o - \frac{\omega_S}{v_{sw}}(r - r_o)$$

where r is the radial distance away from the Sun (which equals r_o at $t = t_o$), ω_S is the angular frequency of the Sun and the solar wind speed $v = v_{sw}$.
[3 marks]

(iv) The angle made by a field line to the radial direction is called the spiral angle and is given by ψ. Show that

$$\tan\psi = \frac{\omega_S(r - r_o)}{v_{sw}} .$$

[3 marks]

(v) For a solar wind speed of $400\,\mathrm{kms^{-1}}$, calculate the value of the spiral angle at Earth orbit, where $1\,\mathrm{AU} = 215\,\text{solar radii} = 1.5 \times 10^8\,\mathrm{km}$. Draw a simple schematic of the spiral magnetic field lines in the equatorial plane, including the shape of the field lines at 1 AU.
[2 marks]

[TOTAL 20 marks]

5. (i) Two different types of plasma flow arise in the Earth's magnetosphere; corotation and convection. Give a brief description of how each of these flows are formed and describe the resultant flow pattern for each.

[4 marks]

(ii) A simple model of the combined corotation/solar wind driven convection (Dungey cycle) flow may be obtained by taking the electrostatic potential in the equatorial plane to be:

$$\Phi(r, \phi) = -\left[E_o r \sin\phi + \frac{\omega_p B_{eq} R_p^3}{r} \right]$$

where E_o is the cross-magnetospheric electric field associated with the Dungey cycle, ω_p the angular frequency of the planetary rotation, B_{eq} the equatorial field strength at the planet's surface, R_p the planet's radius and ϕ the azimuthal angle measured positive anti-clockwise from the noon meridian towards dusk. The gradient vector in polar co-ordinates in a plane is:

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Use this information to find the position of the equatorial stagnation point (R_{sp}, Φ_{sp}) in the flow (that is, where $\mathbf{E} = 0$). Show that the value of the potential at the stagnation point is $\Phi_{sp} = -2E_o R_{sp}$.

[6 marks]

(iii) The location of the stagnation streamline can be found by writing $\Phi(r, \phi) = \Phi_{sp}$. Show that this equation reduces to a quadratic for (r/R_{sp}), and that the solution to this equation is:

$$\frac{r}{R_{sp}} = \frac{\left(1 \pm \sqrt{1 - \sin\phi}\right)}{\sin\phi} .$$

[3 marks]

(iv) Draw a sketch in the equatorial plane showing the combined flow pattern at the Earth (with dusk and noon clearly marked). Point out the stagnation point, stagnation streamline and comment on the resultant flows and on that which is dominant at Earth.

[7 marks]

[TOTAL 20 marks]

6. A cavity, known as the magnetosphere, forms around the Earth as a result of the interaction between the solar wind and the Earth's magnetic field.

(i) Draw an overview sketch of the Earth's magnetosphere in the noon-midnight meridian plane. Show clearly on the sketch and discuss briefly each of the following:

(a) Solar wind flow and bow shock.

(b) Magnetosheath.

(c) Magnetopause.

(d) Earth's magnetic field lines, on dayside and nightside.

(e) Magnetotail and the tail reconnection point.

[6 marks]

(ii) Seen in cross-section, the near Earth geomagnetic tail consists of two D-shaped lobes of oppositely directed magnetic fields, separated by a current sheet in the centre of the tail. If one stands far downtail and looks back towards the Earth, draw a schematic showing a view of the cross-tail current and tail lobe magnetic field lines (of constant field strength $B_T \sim 20\,\mathrm{nT}$). Also show the magnetopause currents via which the tail current closes. [3 marks]

(iii) Assuming the diameter of the tail is $\sim 40\,R_E$ (where the Earth's radius $= R_E = 6400\,\mathrm{km}$), and the average cross-tail electric field associated with the solar wind driven convection cycle is $\sim 2 \times 10^{-4}\ \mathrm{Vm}^{-1}$, one is able to estimate that it takes 3.6 hours for an open flux tube to flow from the northern (or southern) tail lobe magnetopause to the central current sheet. This time is also an estimate for the duration that a given field line remains open during the solar wind driven convection cycle, whilst it is being carried downstream away from the Earth by the solar wind at a velocity of $\sim 400\,\mathrm{km\,s^{-1}}$.

Draw a schematic in the noon-midnight meridian plane showing closed dayside field lines, reconnected open field lines in the cusp region, open field lines flowing downstream and then reconnected field lines in the centre of the magnetotail. Also show the direction of plasma flow in the nightside magnetosphere.

Calculate the length of the geomagnetic tail in R_E. [6 marks]

(iv) Draw a schematic of the view looking down on the north pole of the Earth showing the form of the plasma flow in the ionosphere which is associated with the solar wind driven convection cycle. Mark the local times of noon, dusk, midnight and dawn on the figure. Show circles representing the open flux and closed flux regions. Complete the diagram by requiring the ionospheric plasma to flow around closed paths.

[5 marks]

[TOTAL 20 marks]

SECTION B

7. The upper subphotospheric layers and atmosphere of the Sun can be represented as a plane-parallel layer with constant downward gravitational acceleration **g**, in which the equilibrium pressure p_0 and density ρ_0 are functions only of the vertical height coordinate z. Suppose the layer undergoes perturbations that are small in the sense that quadratic or higher order expressions involving the velocity **u**, the Eulerian pressure perturbation p' or the density perturbation ρ' can be neglected. The Eulerian perturbation to the gravitational acceleration is negligible, as are viscous forces.

(i) Starting from the full momentum equation, derive the linearized momentum equation

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' + \rho' \mathbf{g} ,$$

and write down also the linearized continuity equation and linearized adiabatic energy equation. Show that for adiabatic oscillations

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \left[\rho_0 c^2 \nabla.\mathbf{u} + \rho_0 \mathbf{u}.\mathbf{g}\right] - \nabla.(\rho_0 \mathbf{u})\mathbf{g} ,$$

where c is the adiabatic sound speed (which should be defined). [5 marks]

(ii) Suppose that **u** is a function of x, z, t but is independent of Cartesian coordinate y; in particular, let **u** be proportional to $\exp(i\omega t + ikx)$. Show that **u** has no component in the y-direction, and show furthermore that its x- and z-components (u_x and u_z respectively) satisfy

$$\begin{aligned} -\omega^2 u_x &= ik\left(c^2 \nabla.\mathbf{u} - g u_z\right) \\ -\omega^2 u_z &= \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 c^2 \nabla.\mathbf{u}\right) - g\frac{\partial u_z}{\partial z} + g\nabla.\mathbf{u} . \end{aligned}$$

Demonstrate that it is possible to solve these equations with a velocity **u** for which $\nabla.\mathbf{u} = 0$, provided ω has a particular dependence on k which you should find. Sketch u_z as a function of z. [6 marks]

(iii) Under what circumstances would your solution in part (ii) be a physically acceptable solution as $z \to \pm\infty$? Identify this solution with a class of observed oscillations of the Sun. [3 marks]

(iv) Assuming that the conductivity is effectively infinite, how would your answers to parts (i) and (ii) be modified if the layer were permeated by a uniform equilibrium magnetic field $\mathbf{B_0}$ of strength B in the y-direction? [6 marks]

[TOTAL 20 marks]

8. (i) (a) Give a definition of the *heliospheric current sheet* and provide a brief physical explanation of why it forms. [3 marks]

(b) The surface current density in an idealised equatorial heliospheric current sheet at a heliocentric distance r can be represented by a vector $\mathbf{k}$ with components given by

$$k_r = \frac{2B_{r0}\Omega r_0^2}{\mu_0 v_r r} \qquad and \qquad k_\phi = \frac{2B_r r_0^2}{\mu_0 r^2}$$

in a spherical polar coordinate system, where B_r is the radial component of the magnetic field at r, B_{r0} is the magnetic field at a reference distance r_0, v_r is the solar wind speed, Ω is the solar rotation rate, and μ_0 is the permeability of free space.

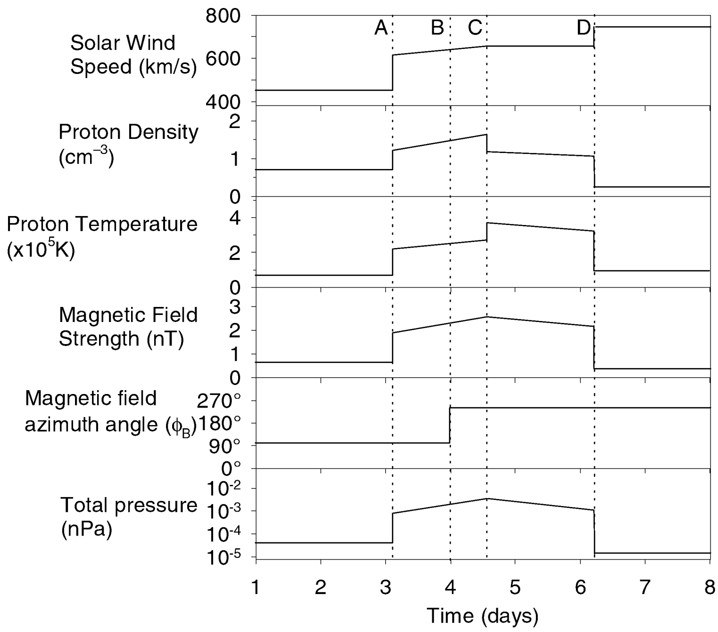
What pattern do these current streamlines describe? Derive an equation for the total current flowing outwards in the heliospheric current sheet. [3 marks]

(c) Is there a net outward current flow from the Sun? If not, where is the current flow balanced? [2 marks]

(ii) The diagram below represents a schematic of a set of signatures often observed in solar wind and magnetic field data from heliospheric spacecraft.

(a) Identify the type of solar wind structure producing the signatures.

(b) Provide a physical explanation for the formation of this structure, in particular identifying the four features marked by the vertical lines A, B, C, D and explaining their physical origin. [12 marks]



[TOTAL 20 marks]

9. There are four main types of interactions which occur between the solar wind and solar system bodies.

(i) List all four types of interaction and describe two of these by detailing the properties of the planetary body involved as well as the properties of the resulting interaction. [6 marks]

(ii) Draw a schematic of the plasma environment of an active comet showing the various boundaries that form. [2 marks]

(iii) For the case of an interaction when a magnetosphere is generated, we assume the planet has an internal magnetic field described as a dipole where:

$$B(r) = B_{eq}\left(\frac{R_p}{r}\right)^3$$

B_{eq} is the field strength at the equator on the planet's surface, R_p is the planet's radius, r is the radial distance from the planet. Write down, without derivation, the resulting total magnetic field just inside the magnetopause boundary by taking account of the effect of the magnetopause layer and by ignoring the interplanetary magnetic field. Explain your answer. [2 marks]

(iv) The Chapman-Ferraro equation yields the magnetopause stagnation distance, R_{MP}, in terms of planetary radii. Derive this equation by assuming pressure balance (between the solar wind plasma and the planetary magnetic field) across the magnetopause boundary. Parameters appearing in your solution should include B_{eq}, μ_o the plasma permeability, m_p the proton mass, n_{sw} the solar wind density and v_{sw} the solar wind velocity. [4 marks]

(v) Draw a schematic of the magnetosphere of Jupiter in the noon-midnight meridian plane, showing the magnetic field lines (and their direction), the various plasma boundaries, plasma regions and the plasma sheet. Without calculation state the value of the stagnation point at Jupiter which results from the equation you derived in (iv). Is this value correct in matching observations and if not what value does one expect? What additional physical effects need to be taken into account to yield a more reasonable value? [6 marks]

[TOTAL 20 marks]

End