2.5 Method of Images

2.5.1 Motivitation

The consequences of the presence of a conductor in the vicinity of a charge distribution or electric fields from other sources arises in many applications, not least of which is the ordinary mirror. Yet the consequence of a mirror, which is nothing more than a highly polished (= flat) conducting surface, is quite simple: it generates an *image*. For reasons that are essentially the same, the action of placing a charge near a conducting surface can also be regarded as adding an *image charge*. This turns out to simplify greatly subsequent analyses, and is known as the Method of Images.

2.5.2 Basic Principles

Consider a charge q placed a distance d above an earthed plane conducting surface, as sketched in Figure 1. Within the conductor, the total electric field must be zero (if it weren't, the free charges in the conductor would move until it were!). At the surface of the conductor, there are free charges that can move tangentially along the surface. Obviously (!?), if there were a tangential component of the electric field there, $\mathbf{E}_{tang}$, the charges on the surface would move and redistribute themselves until $\mathbf{E}_{tang} = \mathbf{0}$. Thus, the resulting configuration must have

$$\boxed{\mathbf{E}_{tang} = \mathbf{0} \text{ at the surface of a conductor}} \quad (1)$$

Now, the electric field is the vector gradient of the electrostatic potential, V, so that $\mathbf{E} = -\nabla V$. So $\mathbf{E}$ is perpendicular to surfaces of constant V. Thus, since the tangential field vanishes at the surface of the conductor, that surface must be a surface of constant V. This is trivial in the case of an earthed conductor (since by definition the potential will be zero), but the argument also holds for an isolated conductor. Thus

$$\boxed{\text{The surface of a conductor is a surface of equipotential}} \quad (2)$$

Expressions (1) and (2) form the basis for understanding the properties of conducting surfaces, and also for constructing solutions using image charges.

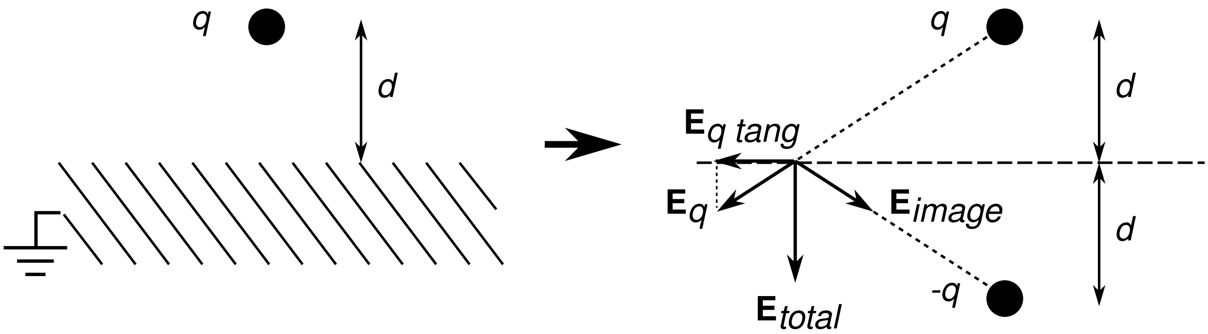

Figure 1: (left) A charge q placed above a conducting surface. (right) Equivalent system with an image charge $-q$ placed at the same distance below the surface. The image cancels out the tangential component of $\mathbf{E}_q$, so that the resulting field in the region above the conductor can be found as the superposition of $\mathbf{E}_q$ and $\mathbf{E}_{image}$.

2.5.3 Image Charges

While it is not at all obvious how to go about finding the redistribution of charge along the surface of our conductor, it is quite easy to find an equivalent system that meets the necessary physical requirements, namely (1). We need something that will cancel the tangential components of the $\mathbf{E}_q$ due to our original charge. If we place an *image charge* of equal magnitude and opposite sign the same distance below the surface, then by superposition the resulting $\mathbf{E} = \mathbf{E}_q + \mathbf{E}_{image}$ will have zero tangential component and twice the normal component at the surface, as Figure 1 clearly shows. **NOTE** that this combination does NOT give the right **E** within the conductor, which we know to be zero. But it does satisfy the boundary conditions imposed on the region above the surface by the presence of the conductor, and hence it does yield the correct **E** above the conductor. The final configuration is shown in Figure 2.

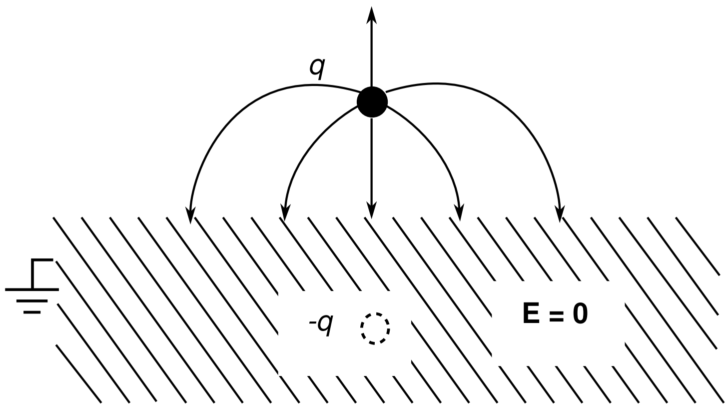

Figure 2: The resulting electric field due to a charge q positioned above an earthed plane conductor. Within the conductor $\mathbf{E} = \mathbf{0}$ while above the conductor the field is dipolar, being the superposition of the field $\mathbf{E}_q$ due to the charge q and that of charge $-q$ which is its image.

You may well recognise the form of the lines of **E**, as they are just those of a dipole. It is also obvious from the fact that this is a dipole configuration that the surface of the conductor is the mid-plane of the dipole. Thus by symmetry the electric field must be normal to the surface there, i.e., it has no tangential component as required by (1).

2.5.4 Surface Charge Density

Let us now revisit the actual physics, which is that the field is actually due to a distribution of charges along the surface of the conductor. What is the charge density there?

This question is now surprisingly easy to answer, since we know the electric field everywhere. We simply apply Gauss's Law to a small pillbox that straddles the surface, as sketched in Figure 3. Only the top surface is threaded by an electric field. This electric field is $\mathbf{E}_{total}$ as shown in Figure 1 and is simply twice the normal component of the electric field $\mathbf{E}_q$ due to the original charge q. Thus Gauss's Law gives:

$$\oint\!\!\!\oint \mathbf{E} \cdot \mathbf{dA} = 2E_{qn}A = \sigma A/\epsilon_o$$

Now $\left|\mathbf{E}_q\right| = q/\left[4\pi\epsilon_o\left(d^2 + r^2\right)\right]$ where r is the distance along the surface from the directly underneath q to the point in question. The normal component has magnitude $\left|\mathbf{E}_q\right| \cos\theta$ with $\cos\theta = d/\sqrt{d^2 + r^2}$

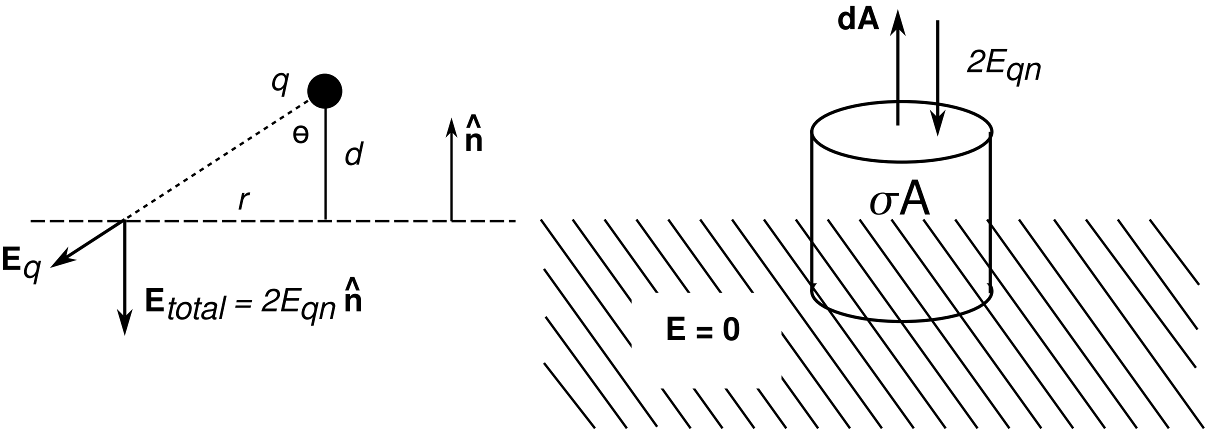

Figure 3: (left) Calculating the normal component of **E** as twice that due to the original charge q and (right) using that in Gauss's Law to determine the charge per unit area σ on the surface of the conductor.

and, for **dA** as shown, E_n is negative, thus

$$\sigma = \frac{-\epsilon_o 2qd}{4\pi\epsilon_o \left(d^2 + r^2\right)^{3/2}} \quad (3)$$

$$= \frac{-qd}{2\pi \left(d^2 + r^2\right)^{3/2}} \quad (4)$$

2.5.5 Other Configurations

While the method is particularly simple for single charges and plane conductors, it can also be applied to more complicated situations. Some of the more common include:

Dipole above a plane can be treated as two single charges. A simple sketch will convince you that in this case the appropriate image is another dipole located the same distance below the conductor, and that the image dipole moment is in the *same* direction as the original one.

Corner conductors are formed by two planes, say horizontal and vertical. A single charge will have a simple image on the opposite side of each plane. But this is not enough to force $\mathbf{E}_{tang}$ to be zero everywhere. Indeed, the image associated with the horizontal plane needs an image behind the vertical plane. It turns out this is the same as the image needed by the second of the original images, so the result is the original charge plus 3 images.

Parallel plane conductors generate an infinite set of images, just like a parallel set of mirrors do. Indeed, two arbitrarily-oriented planes generate an infinite set of images; the corner conductor described above is a special case.

Spheres, cylinders, etc. can also be analysed by the use of images though, just like spherical mirrors, the size of the image is not the same as the original charge.

All these cases and more can be analysed using the two basic principles of conducting surfaces (1) and (2). In most of the curved shapes, it is usually easier to work with the potential and demand that the surface be at a constant potential (zero if earthed).