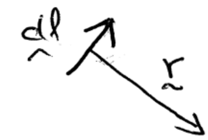
E & M Problem Sheet 7 2008 Solutions

1) Biot & Savart $d\underset{\sim}{B} = \frac{\mu_0}{4\pi} \frac{I \, d\underset{\sim}{l} \times \underset{\sim}{r}}{r^3}$

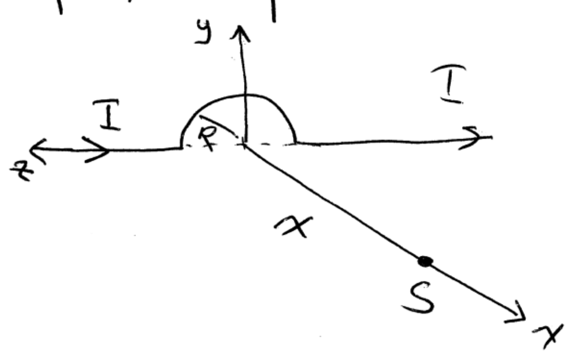
Straight segments have $d\underset{\sim}{l} \parallel \hat{\underset{\sim}{r}}$ so don't contribute. Sections of semicircle have $d\underset{\sim}{l} \times \hat{\underset{\sim}{r}} = dl \, R \, (\widehat{\text{into page}})$ as $d\underset{\sim}{l} \perp r \; \hat{\underset{\sim}{r}}$, so



$$\underset{\sim}{B} = \int_{\text{semicircle}} \frac{\mu_0}{4\pi} \frac{I R \, dl}{R^3} (\widehat{\text{into page}}) = \frac{\mu_0}{4\pi} \frac{I}{R^2} \pi R (\widehat{\text{into page}}) = \underline{\frac{\mu_0 I}{4R} (\widehat{\text{into page}})}$$

This is $\frac{1}{2}$ result for a single loop (see formula with $x=0$) and could be argued directly by symmetry

2) Take hint and solve in pieces.

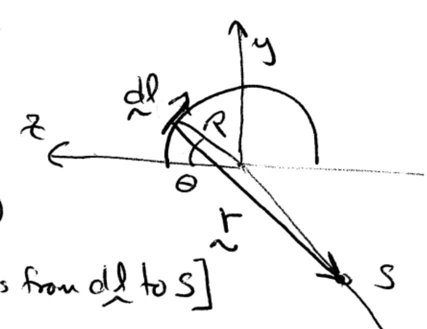


a) ∞ wire gives $\underset{\sim}{B}_1 = \frac{\mu_0 I}{2\pi x} (-\hat{y})$

b) straight segment $2R$ carrying $-I$ gives

$$\underset{\sim}{B}_2 = \frac{\mu_0 I}{4\pi} \frac{2R}{x\sqrt{R^2+x^2}} (+\hat{y})$$

c) For semicircle we set up something similar in CW6. From diagram



$$d\underset{\sim}{l} = R \, d\theta \, \hat{\theta} = R \, d\theta (-\sin\theta \, \hat{z} + \cos\theta \, \hat{y})$$

$$\hat{\underset{\sim}{r}} = x\hat{x} - R\sin\theta \, \hat{y} - R\cos\theta \, \hat{z} \quad [\underset{\sim}{r} \text{ is from } d\underset{\sim}{l} \text{ to } S]$$

Then $d\underset{\sim}{B} = \frac{\mu_0 I}{4\pi} \frac{R \, d\theta (-\sin\theta \, \hat{z} + \cos\theta \, \hat{y}) \times (x\hat{x} - R\sin\theta \, \hat{y} - R\cos\theta \, \hat{z})}{(x^2+R^2)^{3/2}}$

ie $d\underset{\sim}{B} = \frac{\mu_0 I}{4\pi} \frac{R \, d\theta}{(x^2+R^2)^{3/2}} \left[(-R\sin^2\theta - R\cos^2\theta)\hat{x} - x\sin\theta \, \hat{y} - x\cos\theta \, \hat{z} \right]$

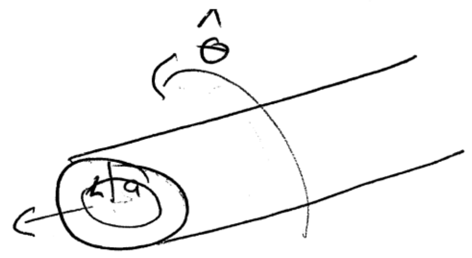
Hence $\underset{\sim}{B}_3 = \int_{\theta=0}^{\pi} d\underset{\sim}{B} = \frac{\mu_0 I}{4\pi} \frac{R}{(x^2+R^2)^{3/2}} \left[-R\pi\hat{x} + x\cos\theta \Big|_0^{\pi} \hat{y} - x\sin\theta \Big|_0^{\pi} \hat{z} \right]$

$$= \frac{\mu_0 I}{4\pi} \frac{R}{(x^2+R^2)^{3/2}} \left[-R\pi\hat{x} - 2x\hat{y} \right]$$

Thus $\underset{\sim}{B} = \underset{\sim}{B}_1 + \underset{\sim}{B}_2 + \underset{\sim}{B}_3 = \underline{\frac{\mu_0 I}{4\pi} \left\{ \frac{-R^2\pi}{(x^2+R^2)^{3/2}} \hat{x} + \left(\frac{-2}{x} + \frac{2R}{x\sqrt{R^2+x^2}} - 2x \right) \hat{y} \right]}$

as $x \to 0$ this reduces to result in prev. question

(3) $J = J_0 e^{r^2/a^2}$



$r<a$ take Amperian circuit

Ampere: $\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 I_{encl} = \mu_0 \iint \underset{\sim}{J} \cdot d\underset{\sim}{A}$

By symm. $\underset{\sim}{B} = B(r)\hat{\theta}$ so with $d\underset{\sim}{l} = r\,d\theta\,\hat{\theta}$:

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = 2\pi r B = \mu_0 \iint \underset{\sim}{J} \cdot d\underset{\sim}{A} = \mu_0 \int_{r=0}^{r}\int_{\theta=0}^{2\pi} J_0 e^{r^2/a^2}\, r\,d\theta\,dr$$

ie $2\pi r B = \mu_0 J_0 2\pi \int_{r=0}^{r} r e^{r^2/a^2} dr = \mu_0 J_0 2\pi \frac{a^2}{2} \int_0^{r^2/a^2} e^{r^2/a^2} d(r^2/a^2)$

$= \mu_0 J_0 \pi a^2 \left[e^{r^2/a^2} - 1\right]$

so $B = \frac{\mu_0 J_0}{2} \frac{a^2}{r} \left[e^{r^2/a^2} - 1\right] \quad r<a$

$r>a$ take Amperian circuit outside. Proceed as above excep $\int_{r=0}^{a}$ only

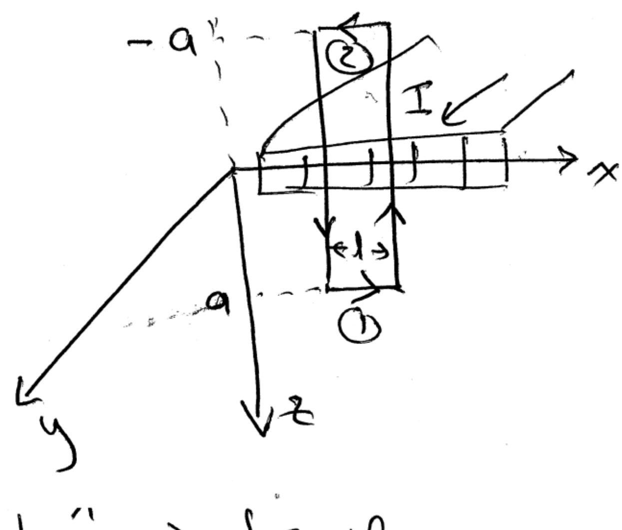
$$2\pi r B = \mu_0 J_0 2\pi \frac{a^2}{2} \int_0^1 e^{(r^2/a^2)} d(r^2/a^2) = \mu_0 J_0 \pi a^2 (e-1)$$

so $\underline{B = \frac{\mu_0 J_0 a^2}{2r}(e-1)} \quad r>a$

(4) No B_y (|| current)

No B_z (Gauss + symmetry)

Symmetry $\Rightarrow B_x(-z) = -B_x(z)$



(a) & (b) Take Amperian circuit as shown. Along sides, $\underset{\sim}{B} = B_y\hat{y}$ ⊥'r $d\underset{\sim}{l} = dz\,\hat{z} \Rightarrow \int \underset{\sim}{B} \cdot d\underset{\sim}{l} = 0$

along (1) $d\underset{\sim}{l} = |dx|\hat{x}$ so $\int \underset{\sim}{B} \cdot d\underset{\sim}{l} = B_x(a)\,l$

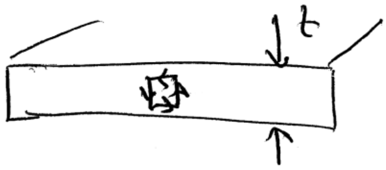
along (2) $d\underset{\sim}{l} = -|dx|\hat{x}$ so $\int \underset{\sim}{B} \cdot d\underset{\sim}{l} = B_x(-a)(-l) = +B_x(a)\,l$ using symmetry.

(4) (cont.)

Ampere: $\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = 2B_x(a)\,l = \mu_0 I_{encl} = \mu_0 n I l$

$\Rightarrow \underline{B_x(a) = \frac{\mu_0 n I}{2}}$ then symmetry $\Rightarrow \underline{B_x(-a) = -\frac{\mu_0 n I}{2}}$

(c) Take Amperian circuit within current sheet; same symmetry. Now in a length l the total current is nIl



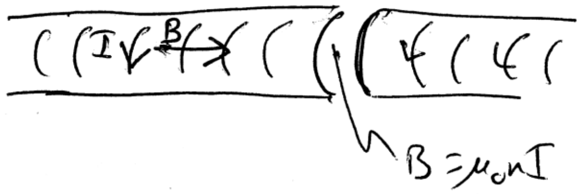
so $J = \frac{I_{total}}{area} = \frac{nIl}{lt} = \frac{nI}{t}$

Ampere's Law: $\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = 2B_x(z)\,l = \mu_0 \frac{nI}{t}(2zl)$

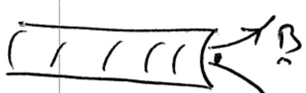
$\Rightarrow \underline{B_x(z) = \mu_0 n I \frac{z}{t}}$ $\quad -\frac{t}{2} < z < t/2$

(5) a) away from end just like ∞ solenoid, ie

$B = \mu_0 n I$

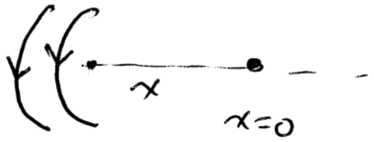


b) with 2 semi-infinite solenoids, get same field where they meet. By symmetry, each contributes same

so  has $B = \frac{\mu_0 n I}{2}$ on axis at end

(6) Single loop distance x gives

$B = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}}$ on axis



In a length dx at x, total current is $I(n dx)$ so

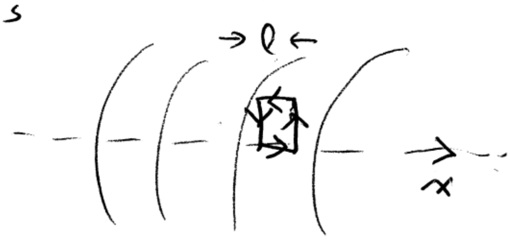
$$dB = \frac{\mu_0 [I n dx] a^2}{2(x^2+a^2)^{3/2}} \Rightarrow B = \frac{\mu_0 I n a^2}{2}\int_{-\infty}^{\infty}(x^2+a^2)^{-3/2}dx$$

let $x = a\tan\theta \Rightarrow x^2+a^2 = a^2(1+\tan^2\theta) = a^2\sec^2\theta$; $dx = a\sec^2\theta\, d\theta$

$$\Rightarrow B = \frac{\mu_0 I n a^2}{2}\int_{-\pi/2}^{\pi/2}\frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = \frac{\mu_0 I n}{2}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{\mu_0 I n}{2}(\sin\theta)\Big|_{-\pi/2}^{\pi/2}$$

ie $\underline{B = \mu_0 I n}$ as required, on the axis

Now construct Amperian circuit that includes a segment along the axis and encloses zero current.



Sides: $\underset{\sim}{B} \perp^r \underset{\sim}{dl}$ by symmetry, so

$$\oint \underset{\sim}{B}\cdot \underset{\sim}{dl} = B_x(r=0)\,l - B_x(r)\,l = \mu_0 I_{encl} = 0$$

since along the top $\underset{\sim}{dl} = -l\hat{x}$

$$\Rightarrow B_x(r) = B_x(0) \Rightarrow B \text{ uniform inside.}$$