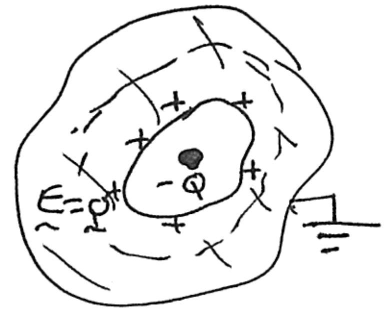
Electricity & Magnetism Prob. Sheet 2 Solutions



(1) a) There must be chg induced on inner surface of metal so that $\underset{\sim}{E}=\underset{\sim}{0}$ within metal. Considering a Gaussian surface within metal & apply Gauss's Law

$$\oiint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 0 = \frac{Q_{encl}}{\epsilon_0} = (-Q + Q_{induced})/\epsilon_0$$

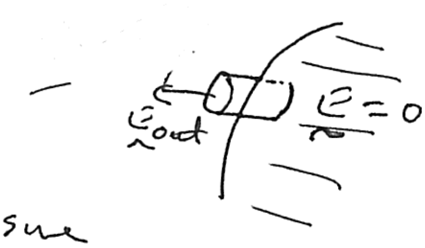
so ~~a~~ total $Q_{ind} = +Q$

b) There is no charge on outside as it is grounded. If there were charge there, then $\Rightarrow \underset{\sim}{E}$ there which would cause charges to move along surface.

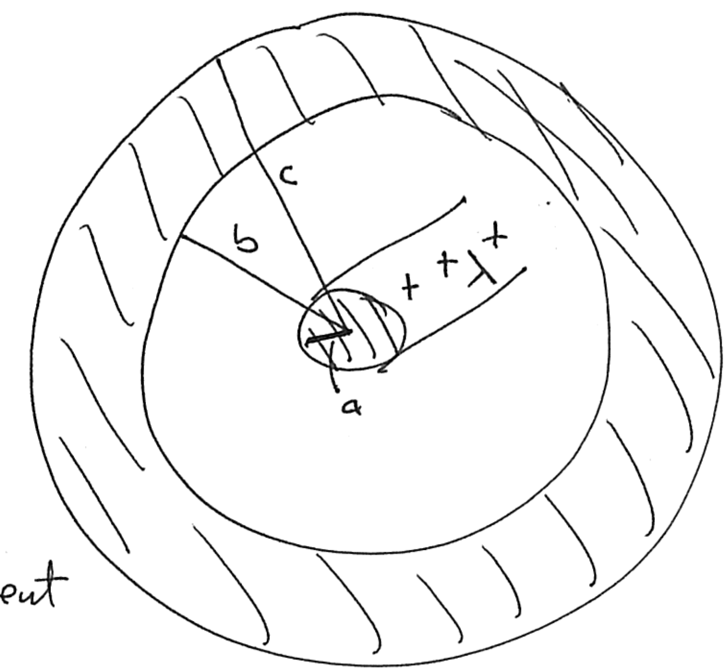
c) There is an $\underset{\sim}{E}$ inside the cavity. Consider surrounding chg $-Q$ with Gaussian surface. Then $\oiint \underset{\sim}{E} \cdot d\underset{\sim}{A} = -Q/\epsilon_0 \neq 0$ $\Rightarrow \underset{\sim}{E} \neq 0$. But there is no $\underset{\sim}{E}$ outside as no charge on surface. To verify, consider Gaussian pillbox at surface.

$\oiint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 0$ as ~~not~~ no chg enclosed.

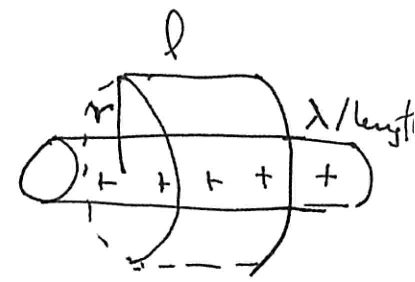
Also $\underset{\sim}{E} = \underset{\sim}{0}$ within metal $\therefore \underset{\sim}{E}_{out} = \underset{\sim}{0}$



d) So someone outside would not measure anything due to the $-Q$ charge, ~~one~~ which is therefore shielded by the grounded conductor. This can't work for gravity as need negative mass to give zero net mass enclosed by a surface.



(2) a) Consider a Gaussian cylinder with radius r $(a<r<b)$ and length l as shown in 2nd diagram.

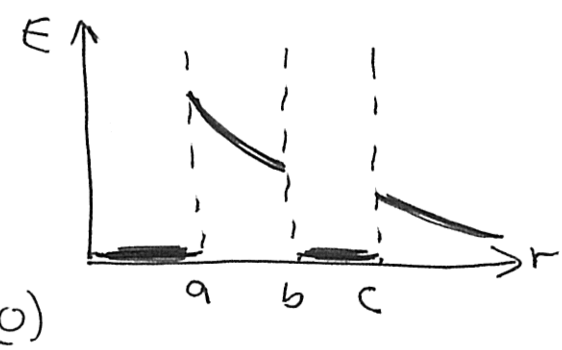


By symmetry, there can only be a radial component of $\underset{\sim}{E}$ which varies only with r, ie $\underset{\sim}{E} = E(r)\hat{\underset{\sim}{r}}$. Then Gauss's Law $\Rightarrow$ $\Phi_E = E 2\pi r l = \dfrac{\lambda l}{\epsilon_0}$

so $$\underset{\sim}{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\hat{\underset{\sim}{r}} \qquad a<r<b$$

b) Since outer cylinder has no net charge, same calculation with Gaussian cylinder $r>c$ also contains λl Coulombs.

$$\therefore \underset{\sim}{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\hat{\underset{\sim}{r}} \qquad r>c$$



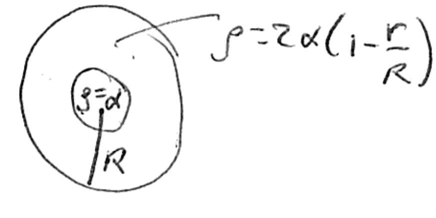
c) Note $\underset{\sim}{E} = \underset{\sim}{0}$ inside conductors

d) Gaussian cylinder for $b<r<c$ must contain zero net charge ($\underset{\sim}{E} = \underset{\sim}{0}$)

Hence: $-\lambda$ per unit length on inner surface

$+\lambda$ " " " " outer surface

(3) a) Total chg $Q = \iiint \rho \, dV$. Using $dV = 4\pi r^2 dr$ in sph. symmetry and given $\rho(r)$ gives



$$Q = 4\pi \int_0^{R/2} \alpha r^2 dr + 4\pi \int_{R/2}^{R} 2\alpha\left(1-\frac{r}{R}\right) r^2 dr$$

$$= 4\pi \frac{\alpha}{3}\left[\frac{R^3}{8} - 0\right] + 8\pi\alpha\left[\left(\frac{R^3}{3} - \frac{R^4}{4R}\right) - \left(\frac{1}{3}\frac{R^3}{8} - \frac{1}{4}\frac{R^4}{16R}\right)\right]$$

$$= \cdots = \frac{5\pi}{8} R^3 \alpha \Rightarrow \underline{\alpha = 8Q/5\pi R^3}$$

b) By symmetry $\underset{\sim}{E} = E(r)\hat{r}$ so take spherical Gaussian surfaces.

Gauss's Law $\Phi_E = \oiint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{Q_{encl}}{\epsilon_0}$

$\Rightarrow$ i) $r < \frac{R}{2}$: $E \, 4\pi r^2 = \frac{1}{\epsilon_0}\int_0^r \alpha \, 4\pi r^2 dr = \frac{\alpha}{\epsilon_0}\frac{4\pi}{3} r^3$

Thus $\underline{E = \frac{\alpha r}{3\epsilon_0} = \frac{8Q}{15\pi\epsilon_0}\frac{r}{R^3}}$ $\quad r < R/2$

ii) $\frac{R}{2} < r < R$: $E \, \cancel{4\pi} r^2 = \frac{1}{\epsilon_0}\left[\int_0^{R/2} \alpha \cancel{4\pi} r^2 dr + \int_{R/2}^{r} 2\alpha\left(1-\frac{r}{R}\right)\cancel{4\pi} r^2 dr\right]$

ie $r^2 E = \frac{\alpha}{\epsilon_0}\left[\frac{1}{3}\frac{R^3}{8} + \frac{2}{3}\left(r^3 - \frac{R^3}{8}\right) - \frac{2}{4R}\left(r^4 - \frac{R^4}{16}\right)\right]$

$= \frac{\alpha}{\epsilon_0}\left[R^3\left(\frac{1}{24} - \frac{2}{24} + \frac{1}{32}\right) + \frac{2}{3}r^3 - \frac{1}{2}\frac{r^4}{R}\right]$

$= \frac{8Q}{5\pi\epsilon_0}\left[-\frac{1}{96} + \frac{2}{3}\frac{r^3}{R^3} - \frac{1}{2}\frac{r^4}{R^4}\right]$

so $E = \frac{8Q}{5\pi\epsilon_0}\left[-\frac{1}{96}\frac{1}{r^2} + \frac{2}{3}\frac{r}{R^3} - \frac{1}{2}\frac{r^2}{R^4}\right]$ $\quad \frac{R}{2} < r < R$

iii) $r > R$: $E \, 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow \underline{E = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}}$ $\quad r > R$

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3) (cont) Check that E is continuous at $r=\frac{R}{2}$ and $r=R$:

$$\underline{E(r=\tfrac{R}{2}-\epsilon) \overset{?}{=} E(r=\tfrac{R}{2}+\epsilon)} \Rightarrow$$

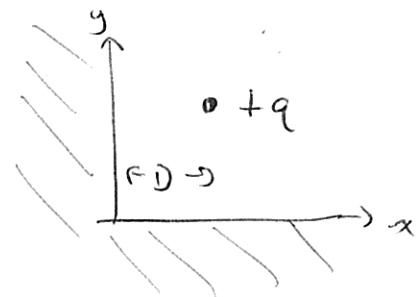
$$\frac{\cancel{8Q}}{15\cancel{\pi\epsilon_0}}\frac{\cancel{R}}{2R^{\cancel{3}}} \overset{?}{=} \frac{\cancel{8Q}}{5\cancel{\pi\epsilon_0}}\left[\frac{-1}{96}\frac{4}{R^2} + \frac{2}{3}\frac{\cancel{R}}{2R^{\cancel{3}}} - \frac{1}{2}\frac{R^{\cancel{2}}}{4R^{\cancel{4}}}\right]$$

ie $$\frac{1}{30} \overset{?}{=} \frac{1}{5}\left[\frac{-1}{24} + \frac{2}{6} - \frac{1}{8}\right] = \frac{1}{5\times 24}(-1+8-3) = \frac{1}{30} \quad ✓$$

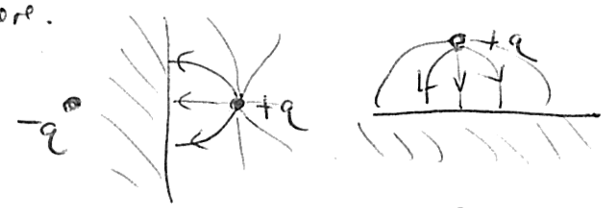
Sim at $r=R\pm\epsilon$:

$$\frac{8\cancel{Q}}{5\cancel{\pi\epsilon_0}}\left[-\frac{1}{96}\frac{1}{\cancel{R^2}} + \frac{2}{3}\frac{1}{\cancel{R^2}} - \frac{1}{2}\frac{1}{\cancel{R^2}}\right] \overset{?}{=} \frac{\cancel{Q}}{4\cancel{\pi\epsilon_0}}\frac{1}{\cancel{R^2}} \quad \text{ie}$$

$$\frac{8}{5}\frac{1}{96}\left[-1+64-48\right] = \frac{1}{5\times 12}(15) = \frac{1}{4} \quad ✓$$

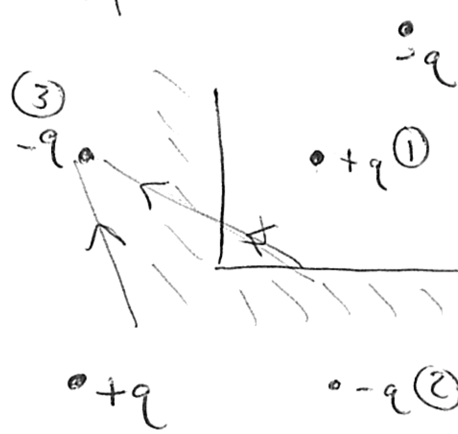


4) Note: In general, charges move in a conductor until $\underset{\sim}{E} \to 0$, which implies that $\underset{\sim}{E}$ ⊥'r surface of conductor. This can be mimicked by use of a suitable image charge that cancels $\underset{\sim}{E}$ tangential there.



Simple plane leads to image of opp charge and same distance, as shown.

If we put these two together, as in 3rd sketch and look along x-axis, can see that although ① & ② combine to make $\underset{\sim}{E}$ ⊥'r there, we are left with tangential component from ③. To solve this, we need the image of ③ in the lower half plane of magnitude $+q$. This will cancel the tangential component from ③.



It is also the image in the left side of ②, and so will cancel the tangential component of ② along the y-axis. So the answer to "?" is $+q$. This is a positive image of the original charge. A corner mirror works the same way: inverted/reversed images for each plane, and a true image in the corner.