4.3 Dielectrics

4.3.1 Motivitation

Young & Freedman devote Chapter 24 to a discussion of capacitors and dielectrics. That discussion is quite complete. It defines the general concepts of capacitance as the capacity to store charge ($C = Q/V$; the higher the capacitance the more charge is stored for a given voltage). Various examples of plane parallel plate capacitors, spherical capacitors, etc., are considered, and the relationship between the configuration, the electric field, the electric potential, and also the energy stored in the system are all carefully done.

They then introduce in Sections 24.4 and 24.6 a dielectric medium into the problem. They show how some media, in the presence of an external electric field, become *polarised*, resulting in some *induced charge* appearing at their surface, and hence *reducing* the electric field within them. Since the potential V is the line integral of the electric field, the potential is likewise reduced. Thus, for a given charge Q, a capacitor with a dielectric will have a lower voltage, and hence a higher capacitance.

The purpose of this brief handout is NOT to replicate the material in the book. However, there are many ways of looking at this subject, and equally many bits of notation. Below, I try to relate all of these and cast them in terms of the notation used in the book. Table 1 summarises the main results and the relationship between the various quantities. You don't need to know most of this material, but it might be helpful now or in the future.

4.3.2 Dielectric Basics

A dielectric is a material in which the molecules can be polarised in response to an externally applied electric field. The basic concepts are depicted in Figure 1, which shows a parallel plate capacitor charged with a free charge density σ_o that in vacuum gives rise to an electric field $\mathbf{E}_o = \sigma_o/\epsilon_o$ in the capacitor. If a dielectric medium is inserted in the plates, the external field causes the dipolar molecules within the medium to polarise, with the + ends attracted to the $-\sigma_o$ side of the capacitor and the − ends attracted to the $+\sigma_o$ plate. This results in a charge $\pm\sigma_i$ appearing at the edges of the dielectric; this is said to be the *induced* charge density. Since the molecules are not totally free, the extent to which the dielectric medium polarises depends on its properties and is proportional to the external electric field.

The induced charge density $\pm\sigma_i$ gives rise to a polarisation electric field $\mathbf{E}_{pol}$ which acts in the opposite direction to $\mathbf{E}_o$, leading to a reduction in the electric field inside the capacitor from its vacuum value $\mathbf{E}_o$ to $\mathbf{E} = \mathbf{E}_o/\epsilon_r$ where ϵ_r is the *relative permittivity* of the dielectric.

4.3.3 Formulation

It is straightforward now to formulate all these concepts into a set of mathematical relations. Firstly, we can relate the various electric fields in the problem. As described above, the field inbetween the plates is the (vector) sum of the vacuum field and the polarisation electric field:

$$\mathbf{E} = \mathbf{E}_o + \mathbf{E}_{pol} \tag{1}$$

Since a pair of charge sheets $\pm\sigma$ give rise to an electric field σ/ϵ_o, we can re-write (1) in terms of the charge densities:

$$\mathbf{E} = \frac{\sigma_o - \sigma_i}{\epsilon_o}\hat{\mathbf{x}} \tag{2}$$

where $\hat{\mathbf{x}}$ points to the right in Figure 1. You can use this equation to find the size of the induced charge density.

At the molecular level, the individual molecules have dipole moments $\mathbf{p} \equiv q\mathbf{d}$ which point from the negative end to the positive end of the molecule. So the dipole moments tend to point from left to right ($-\sigma_i$ to $+\sigma_i$) in Figure 1. The induced charge densities $\pm\sigma_i$ are just the ends of the molecules at the edge of the slab. We can add up all the little dipoles to define a *polarisation* $\mathbf{P}$ by

$$\mathbf{P} = \sum_{molecules\ i} N_i < \mathbf{p} > \tag{3}$$

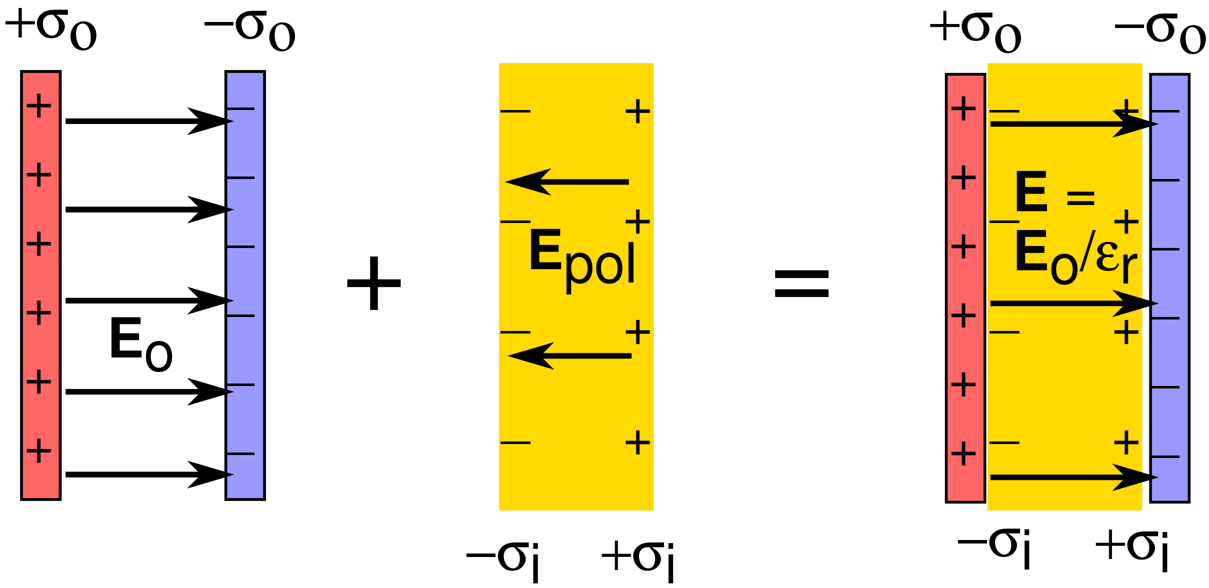

Figure 1: (left) Vacuum capacitor with a free charge σ_o leading to a vacuum electric field $\mathbf{E}_o$. When a dielectric is inserted within the space, it polarises, leading to a surface charge $\pm\sigma_i$ being induced at its edges. The resulting polarisation electric field $\mathbf{E}_{pol}$ reduces the total electric field within the capacitor by a factor ϵ_r, the relative permittivity of the material.

where N_i is the number of dipoles per unit volume. Within the dielectric, the positive and negative ends of the dipoles cancel one another, so **P** only manifests itself for those molecules within *d* or so of the edge. Then Equation 3 counts up that charge, so we have

$$\mathbf{P} = -\epsilon_o \mathbf{E}_{pol} = -\sigma_i \hat{\mathbf{x}} \tag{4}$$

where the minus sign enters because the **p**'s point from negative to positive ends of each dipole, while electric fields point from positive to negative charges.

The extent to which all the dipoles line up depends on the strength of the electric field **E** within the dielectric, i.e. $\mathbf{P} \propto \mathbf{E}$ so let's write

$$\mathbf{P} \equiv \epsilon_o \chi_e \mathbf{E} \tag{5}$$

where χ_e is the susceptibility of the medium to be polarised.

4.3.4 Electric Displacement D

The problem with all the above treatment is that it requires us to look at the microscopic details of the dielectric. In practice, when we charge up a capacitor is it easy to measure the charging current and hence measure the amount of *free* charge σ_o we have put on the plates. Is there a way to hide the details of the dielectric? You bet.

Let's define a vector **D** that will operate in Gauss's Law and can be related to the free charges of the system, i.e.,

$$\oiint \mathbf{D} \cdot d\mathbf{A} = Q_{free} \tag{6}$$

We have absorbed the ϵ_o into **D**, which is known as the Displacement Vector. In our case of a parallel plate capacitor, we know that $Q_{free} = \sigma_o A = \epsilon_o |\mathbf{E}_o| A$. Putting this all into Equations (1-5) leads to

$$\begin{aligned} \mathbf{D} &= \epsilon_o \mathbf{E}_o && (7)\\ &= \epsilon_o \left(\mathbf{E} - \mathbf{E}_{pol}\right) && (8)\\ &= \epsilon_o \mathbf{E} + \mathbf{P} && (9)\\ &= \epsilon_o (1 + \chi_e)\, \mathbf{E} && (10)\\ &= \epsilon_o \epsilon_r \mathbf{E} && (11) \end{aligned}$$

so that the relative permittivity ϵ_r is 1 plus the susceptibility. Equations 7–11 are a translation table amongst the various descriptions of the system. If the medium is not susceptible to being polarised, $\chi_e = 0$, $\epsilon_r = 1$, $\mathbf{D} = \epsilon_o \mathbf{E}$ and, by virtue of Gauss's Law (6), $\mathbf{E} = \mathbf{E}_o$ and is due to the free charge on the plates.

Table 1: Quantities, symbols, definitions, and main results

Quantity	Y&F (or def'n)	Description
ϵ_r	K	Dielectric constant
ϵ	$K\epsilon_o$	Permittivity (we won't use this, but you might see it elsewhere)
χ_e	$K-1 \equiv \epsilon_r - 1$	Susceptibility
$\mathbf{D}$	$\epsilon_o K\mathbf{E} \equiv \epsilon_o\epsilon_r\mathbf{E}$	Displacement vector; this is our primary addition to Y&F, and an important one
$\mathbf{P}$	$\sum_{molecules\ i} N_i < \mathbf{p} >$	Polarisation due to molecular dipole moments **p**
$\mathbf{P}$	$-\epsilon_o\mathbf{E}_{pol}$	Defines **P** in terms of the electric field due to the polarised dipoles
$\mathbf{P}$	$\epsilon_o\chi_e\mathbf{E}$	Defines susceptibility
$\mathbf{D}$	$\epsilon_o\mathbf{E}+\mathbf{P}$	Relates **D**, **E**, and **P**
$\oiint \mathbf{D}\cdot\mathbf{dA} = Q_{free}$	$\oiint \epsilon_o K\mathbf{E}\cdot\mathbf{dA} = Q_{free}$	Gauss's Law

4.3.5 Dielectrics and Capacitors

Since the electric field is reduced by ϵ_r when a dielectric is present, the voltage V between the plates is also reduced for the same free charge $\sigma_o A = Q_{free}$ from V_o to $V = V_o/\epsilon_r$. Thus

$$\begin{aligned} C &= \frac{Q_{free}}{V} = \frac{Q_{free}}{V_o/\epsilon_r} \\ &= \epsilon_r\frac{Q_{free}}{V_o} \equiv \epsilon_r C_o = \epsilon_r\epsilon_o\frac{A}{d} \end{aligned} \qquad (12)$$

(the d here is the separation between the plates of the capacitor).

The electric energy can be calculated by charging the capacitor, starting from zero charge and voltage, and transfering charge from one plate to the other. The result shows that the energy density within the plates can be written

$$u_E = \frac{1}{2}\epsilon_o\epsilon_r E^2 \qquad (13)$$

Thus u_E is larger than the quantity $\frac{1}{2}\epsilon_o E^2$ one might have expected from just considering the electric field itself. This is essentially because some energy is required to polarise the dielectric, and so some energy is stored in the dipole alignment. However, since $E = E_o/\epsilon_r$, the total energy is less in the case of a dielectric than in a vacuum capacitor; it takes less total energy to store the same charge - so it's a better capacitor.

4.3.6 Epilogue

The basic operation of a dielectric is quite simple, and illustrated in Figure 1. Unfortunately, there are many ways of looking at this problem, which introduce a plethora of symbols and definitions, most of which are summarised in Table 1. Interestingly, it is possible to pass from the vacuum form of an equation (e.g., that for the electric field energy density) to the form in the presence of a dielectric merely by replacing ϵ_o (the permittivity of a vacuum) by $\epsilon_r\epsilon_o \equiv \epsilon$ (the permittivity of the medium).

For our purposes, there are really only a couple of things you need to know:

1. Dielectrics reduce the electric field and voltage by a factor of $1/\epsilon_r$ from their vacuum values.

2. The displacement vector $\mathbf{D} \equiv \epsilon_o\epsilon_r\mathbf{E}$

3. $\oiint \mathbf{D}\cdot\mathbf{dA} = Q_{free}$ is Guass's Law and relates the free charges to the displacement vector.

The rest of the material here is for your background, and for you to consult when you run across a book or problem that uses another approach.

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