Imperial College London

BSc/MSci EXAMINATION June 2009

This paper is also taken for the relevant Examination for the Associateship

ELECTRICITY & MAGNETISM

For First-Year Physics Students

Monday, 8 June 2009: 14:00-16:00

Answer both questions in Section A and TWO questions from Section B.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Two charges $q_1 = 1.0\,\text{C}$ and $q_2 = -2.0\,\text{C}$ are located along the x−axis at $x = -1.0\,\text{m}$ and $x = +1.0\,\text{m}$ respectively. Calculate the force on a third charge $q_3 = -1.0\,\text{C}$ located along the y−axis at $y = 2.0\,\text{m}$. [7 marks]

(ii) An infinitely long cylinder of radius a carries a total charge λ per unit length distributed uniformly throughout its volume. It is surrounded by an earthed concentric cylindrical conducting shell of radius $b > a$.

(a) Find the electric field $\boldsymbol{E}$ and the electrostatic potential V for $a < r < b$.

(b) Find the capacitance per unit length C of this cylindrical capacitor.

(c) If the space $a < r < b$ is filled with a dielectric, will C increase or decrease? Provide some explanation for your answer, but do not evaluate the capacitance.

[9 marks]

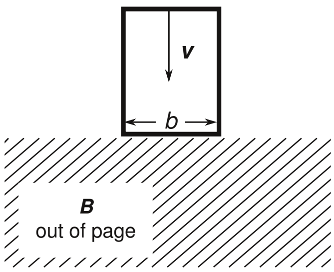
(iii) An infinitely long cylinder of radius a carrying a total charge λ per unit length distributed uniformly throughout its volume is moving along the axial direction $\hat{\boldsymbol{z}}$ at a speed v.

(a) Show that there is a current density $\boldsymbol{j} = \lambda v/(\pi a^2)\,\hat{\boldsymbol{z}}$ in the region $r < a$.

(b) Hence find the magnetic field for the regions $r < a$ and $r > a$.

[7 marks]

(iv) A conducting rectangular loop of width b is inserted into a region permeated by a uniform magnetic field $\boldsymbol{B}$, as sketched in the diagram below. The loop has a total electrical resistance R. At time $t = 0$ the leading edge of the loop begins to enter the magnetic field region. Consider only times for which the loop is NOT fully immersed in the magnetic field region.



(a) State Faraday's Law of Induction, providing an explanation of the various terms and symbols that appear in it.

(b) Show that the magnitude of the current $\mathcal{I}$ in the loop is given by

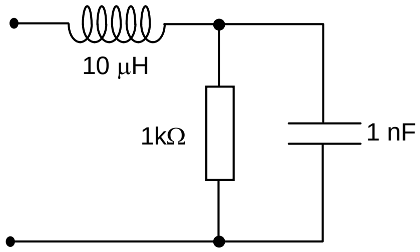
$$|\mathcal{I}| = \frac{Bvb}{R}$$

and indicate on a simple sketch the direction of $\mathcal{I}$ in relation to $\boldsymbol{B}$.

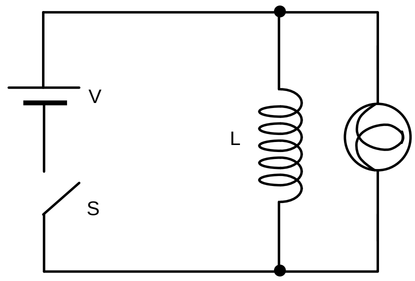
[6 marks]

[Total 29 marks]

2. (i) Calculate the impedance of the circuit shown below at 1 MHz. [3 marks]

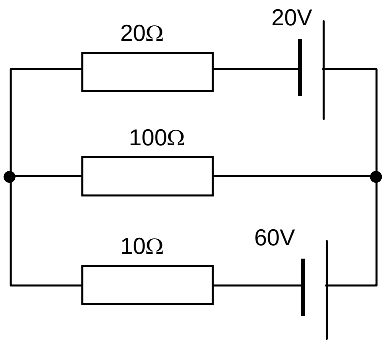


(ii) The circuit shown below consists of an inductor in parallel with a light bulb connected via a switch to a battery. Initially the switch is open. Describe how the intensity of the light behaves when (a) switch S is closed and then, much later, (b) opened. [2 marks]



(iii) A linear source has an open circuit voltage $10\angle 0^\circ$ volts and an output impedance $600 + j100\,\Omega$. Calculate the short circuit current and express this in polar form. [3 marks]

(iv) For the circuit shown below calculate the current in each resistor. [4 marks]



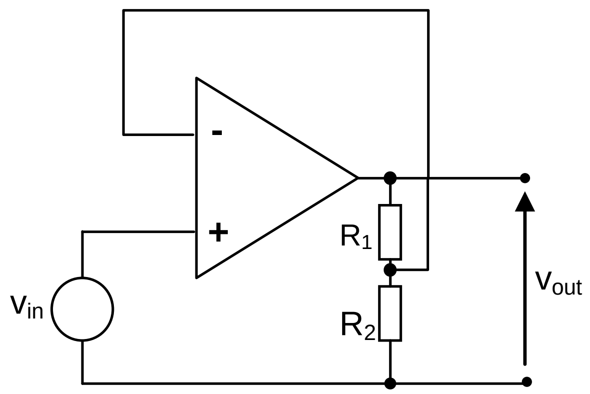
(v) Explain what is meant by the drift velocity, v_d of electrons in a metal wire. Show, by considering a small section of the wire of cross section A, that the current is given by $I = nqv_dA$ where n is the electron density and q is the electronic charge. Hence find the drift velocity of electrons in an aluminium wire of diameter 2.25 mm which carries a current of 5 A. [You may assume that each Al atom contributes one free electron and that the density and the atomic mass of Al is 2.7 g/cm^3 and 26.98 g/mol. respectively] [4 marks]

(vi) The operational amplifier in the circuit shown below has an open-loop gain given by $10^4/(1 + jf/10)$.

(a) Sketch this on a Bode plot with appropriate labels and scale.

(b) Calculate the closed loop low frequency gain and the bandwidth if $R_1 = 95\,k\Omega$ and $R_2 = 5\,k\Omega$. Include the closed loop gain on your sketch.

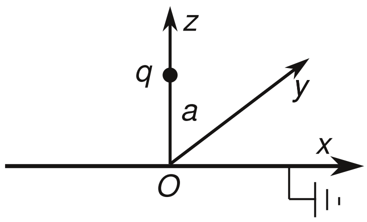
[5 marks]



[Total 21 marks]

SECTION B

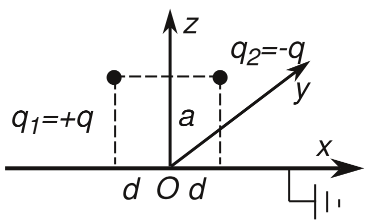
3. (i) An earthed plane conductor occupies the $x - y$ plane. A charge $q > 0$ is placed along the z-axis at $z = +a$ above the plane. (see sketch).



(a) What conditions does the presence of the conductor impose on the electric field $\boldsymbol{E}$ and electrostatic potential V? [4 marks]

(b) Show that an image charge $Q = -q$ placed at a position $\boldsymbol{R} = (0, 0, -a)$ will replicate these conditions. [5 marks]

(ii) The charge $q_1 \equiv q$ is now moved to a position $\boldsymbol{r}_1 = (-d, 0, a)$ (i.e., displaced in the $-\hat{\boldsymbol{x}}$ direction by a distance d). A second charge $q_2 = -q$ is placed at position $\boldsymbol{r}_2 = (+d, 0, a)$. (see sketch below).



(a) Show that in this case there will be two image charges $Q_{1,2}$. Find their locations $\boldsymbol{R}_{1,2}$. [4 marks]

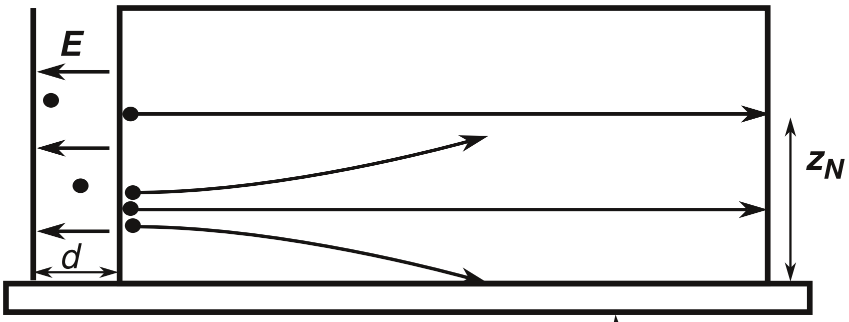
(b) Show that at an arbitrary point $P = (x, y, 0^+)$ just above the surface of the conductor, the $\hat{\boldsymbol{z}}$-component of the electric field is given by

$$E_z = \frac{qa}{2\pi\epsilon_o}\left\{\frac{-1}{\left[(x+d)^2+y^2+a^2\right]^{3/2}}+\frac{1}{\left[(x-d)^2+y^2+a^2\right]^{3/2}}\right\}$$ [8 marks]

(c) Hence or otherwise find the charge per unit area, $\sigma(x, y)$ everywhere along the surface of the conductor. [4 marks]

[Total 25 marks]

4. In a variant of Millikan's famous oil drop experiment, Professor Vectorious attempts to measure the charge of an electron using the magnetic field associated with a straight horizontal wire carrying a current $\mathcal{I}$. Drops of oil of mass m are inserted at rest at the left end of the apparatus, and become charged due to ionising radiation. Let the charge of a drop be $-Ne$ where e is the magnitude of the charge on a single electron. Once charged, the drops are accelerated by a uniform horizontal electric field $\boldsymbol{E}$ for a distance d to a non-relativistic velocity v ($\ll c$). Their initial velocity on passing from the electric field region into the remainder of the apparatus, shown as the large rectangle in the diagram below, is purely horizontal. Thereafter, their motion is confined to the vertical half-plane bounded on the bottom by the wire and they experience the combination of gravitational and magnetic forces. This configuration is shown in the following sketch.



(i) Show that on exiting the electric field region, an oil drop has a speed $v = \sqrt{2NeEd/m}$. [5 marks]

(ii) Show that in the vertical plane containing the wire, the magnetic field is perpendicular to this plane and varies inversely with distance from the wire. [5 marks]

(iii) Write down the equation of motion for the drop and show that if the vertical component of velocity is to remain zero the drop must be a distance

$$z_N = \frac{Nev\mu_o\mathcal{I}}{2\pi mg}$$

from the position of the wire. Indicate clearly on a sketch the direction of $\mathcal{I}$ needed to satisfy this condition and the resulting $\boldsymbol{B}$. [5 marks]

(iv) Hence by combining your results from (i) and (iii) show that there are discrete vertical positions z_N where drops with N electronic charges will pass through horizontally given by

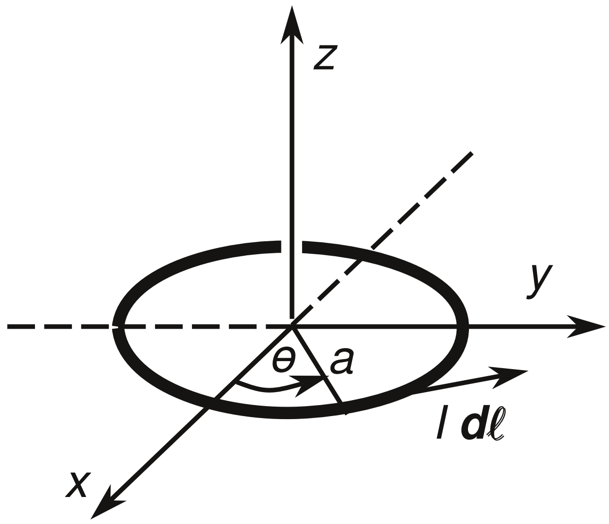
$$z_N = (Ne)^{3/2}/\alpha$$

where $\alpha = \pi g\sqrt{2m^3}/(\mu_o\mathcal{I}\sqrt{Ed})$ [5 marks]

(v) Assuming the modern value for e, for N=2 calculate the expected vertical position z_2 if $m = 10^{-19}$ kg, $\mathcal{I}$ = 100 A, and Ed = 1000 V. [5 marks]

[Total 25 marks]

5. (i) A circular loop of wire of radius a lies in the $x-y$ plane centred at the origin. It is immersed in a uniform magnetic field $\boldsymbol{B} = (0, B_y, B_z)$. A current $\mathcal{I}$ is flowing in the loop in a right-handed sense around the positive $\hat{\boldsymbol{z}}$ direction, as shown below.



(a) Find the magnetic flux ϕ_B through the loop. [4 marks]

(b) Consider an element $\boldsymbol{d\ell} \equiv a\, d\theta\, \hat{\boldsymbol{\theta}}$ with $\hat{\boldsymbol{\theta}} = -\sin\theta\, \hat{\boldsymbol{x}} + \cos\theta\, \hat{\boldsymbol{y}}$ being the usual plane polar unit vector. Show that the magnetic force on this element is given by

$$\boldsymbol{dF} = \mathcal{I}\, a\, d\theta \Big(B_z \cos\theta\, \hat{\boldsymbol{x}} + B_z \sin\theta\, \hat{\boldsymbol{y}} - B_y \sin\theta\, \hat{\boldsymbol{z}} \Big)$$ [4 marks]

(c) Hence or otherwise show that there is no net force on the loop as a whole. [4 marks]

(d) Find the torque $\boldsymbol{d\tau}$ about the x-axis related to the force $\boldsymbol{dF}$ and hence show that there is a net torque $\boldsymbol{\tau}$ on the whole loop given by

$$\boldsymbol{\tau} = -\mathcal{I}\pi B_y a^2\, \hat{\boldsymbol{x}}$$ [5 marks]

(ii) Consider now a loop rotating about the x-axis at an angular frequency ω in the presence of a uniform magnetic field $\boldsymbol{B} = B_o\, \hat{\boldsymbol{z}}$. A current of magnitude $\mathcal{I}$ is maintained in the loop by an external driving circuit. This problem is equivalent to that in part (i) of this question when viewed in a rotating frame, i.e., by substituting $B_z = B_o \cos\omega t$ and $B_y = B_o \sin\omega t$ into your previous results.

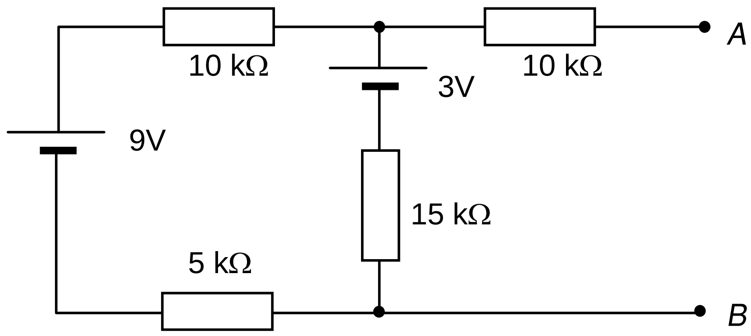
(a) Write down the (time-dependent) torque under these circumstances. [4 marks]

(b) Given that the power delivered by a torque of magnitude τ is $\tau\omega$, calculate the average power delivered by this torque during the interval $0 < \omega t < \pi$. [4 marks]

[Total 25 marks]

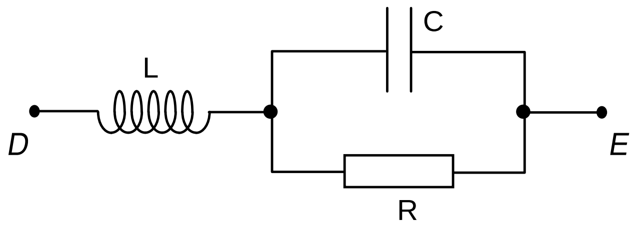
6. (i) Find the Thévenin equivalent between points A and B of the circuit shown below.

[12 marks]



(ii) Find an expression for the impedance of the resonant circuit shown below between points D and E. State the condition for resonance and hence find an equation for the resonant frequency ω_o. Evaluate ω_o for the component values $L = 10\mu\,\mathrm{H}$, $C = 10\,\mathrm{nF}$ and $R = 100\,\Omega$. Which of these components dissipates energy?

[8 marks]



(iii) Draw a circuit diagram for a unity gain buffer and show how this circuit acquires this name. What are the characteristics of this circuit that make it useful?

[5 marks]

[Total 25 marks]