

UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2007

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ELECTRICITY & MAGNETISM

For First-Year Physics Students

Monday 4th June 2007: 14.00 to 16.00

Answer ALL parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Three identical charges $+q$ are arranged in an equilateral triangle of side a . Find the energy required to assemble this configuration, given the convention that the energy is zero when the charges are all separated at infinity. [6 marks]

- (ii) A cylindrical capacitor is formed from two concentric conducting cylindrical shells of radii a and $b > a$. A charge per unit length $+\lambda$ is placed on the inner cylinder, with a charge per unit length $-\lambda$ on the outer cylinder.

- (a) Use Gauss's Law to find the electric field, and potential V , for $r < a$, $a < r < b$, and $r > b$ where r is the radial distance from the centre. [5 marks]

- (b) Hence show that this configuration has a capacitance per unit length given by

$$C = \frac{2\pi\epsilon_0}{\ln(b/a)}. \quad [3 \text{ marks}]$$

- (iii) A solenoid of radius a and infinite length has n turns of wire per unit length carrying an electrical current I .

- (a) Use Ampere's Law to show that the magnetic field within the solenoid is uniform with magnitude $B = \mu_0 n I$. [5 marks]

- (b) Hence, recalling that the self-inductance of a system is defined as the sum of the magnetic flux through the individual loops making up the circuit divided by the current, show that this solenoid has an inductance per unit length L given by

$$L = \mu_0 \pi a^2 n^2. \quad [3 \text{ marks}]$$

- (iv) A circular loop of wire of radius 1 m is immersed in a uniform magnetic field with magnitude that varies according to

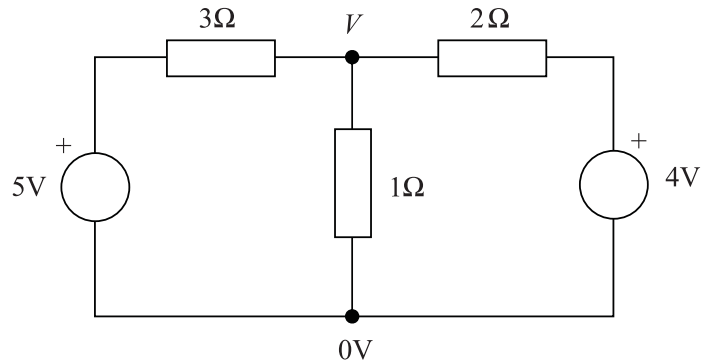
$$B(t) = 10^{-4} \left(1 - \frac{t}{100} \right) \text{ T},$$

where t is the time in seconds. The loop remains oriented such that its plane is perpendicular to the magnetic field. If the resistance of the loop is $R = 100 \Omega$, find the current induced in the wire for $0 < t < 100$ s, indicating on a sketch the direction of this current in relation to the direction of the magnetic field. [7 marks]

[TOTAL 29 marks]

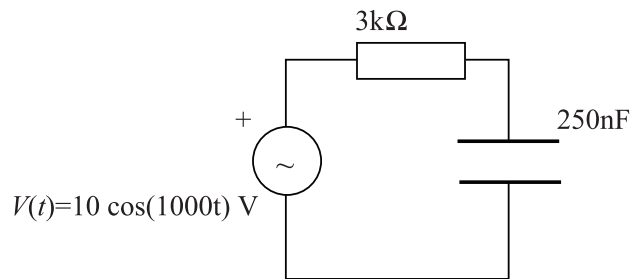
2. (i) State Kirchhoff's current and voltage laws as applied to electrical circuits.

Using node voltage analysis, or otherwise, calculate the unknown voltage, V , in the circuit below and hence determine the current flowing in each of the three resistors.



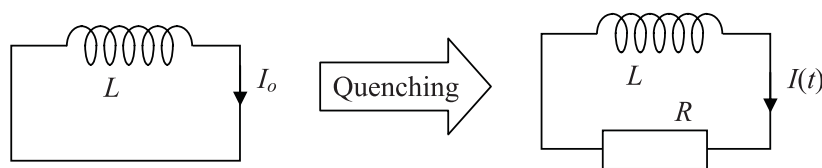
[7 marks]

- (ii) Write down the complex impedance of the individual components and their combination in the circuit shown below. Hence find the current flowing in the circuit as a function of time.



[7 marks]

- (iii) A superconducting magnet can be modelled as a coil of inductance $L = 10\text{H}$, carrying a current of $I_o = 50\text{A}$ in a closed loop. What is the energy stored in the inductor?



The magnet coil suddenly quenches (it accidentally warms up, losing its superconductivity) and acquires a series resistance of value $R = 2.5\ \Omega$.

Describe and plot (without a full derivation) the subsequent evolution of the current in the magnet as a function of time.

What is the peak power dissipated in the magnet (the resistance R) as it quenches?

[7 marks]

[TOTAL 21 marks]

SECTION B

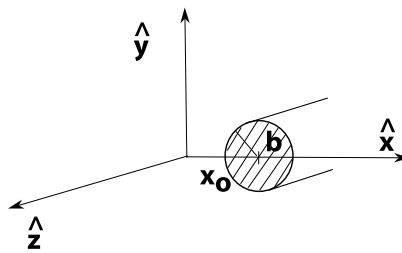
3. (i) A cylinder of radius a carries a uniform charge per unit volume ρ_1 . The axis of the cylinder coincides with the \hat{z} axis. Find the electric field inside the cylinder and hence show that the electric potential is given by

$$V_1 = -\frac{\rho_1 r^2}{4\epsilon_0} + V_{10} ,$$

where V_{10} is a constant and r is the distance from the \hat{z} -axis.

[9 marks]

- (ii) Hence deduce that the potential within a cylinder of radius b carrying a uniform charge density ρ_2 and displaced a distance x_0 along the x -axis as shown below



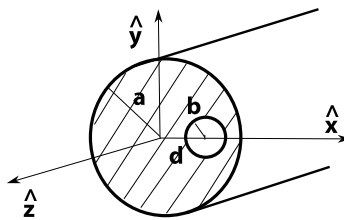
is given by

$$V_2 = -\frac{\rho_2}{4\epsilon_0} |\mathbf{r} - x_0 \hat{\mathbf{x}}|^2 + V_{20} ,$$

where V_{20} is a constant.

[4 marks]

- (iii) Consider now a cylinder carrying a uniform charge density ρ of radius a out of which a hole of radius $b < a$ is cut, the centre of which is a distance d from the centre along the x -axis, as shown in the accompanying diagram.



Show that within the hole the electric field is uniform and given by

$$\mathbf{E} = +\frac{\rho d}{2\epsilon_0} \hat{\mathbf{x}} .$$

[6 marks]

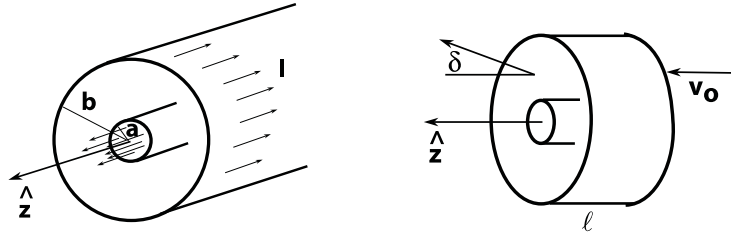
- (iv) The hole in the cylinder is now filled with a dielectric with dielectric constant ϵ_r . Show that the electric field within the dielectric in this case is no longer uniform, but given by

$$\mathbf{E} = +\frac{\rho}{2\epsilon_0} \left\{ \left[x \left(1 - \frac{1}{\epsilon_r} \right) + \frac{d}{\epsilon_r} \right] \hat{\mathbf{x}} + y \left(1 - \frac{1}{\epsilon_r} \right) \hat{\mathbf{y}} \right\} .$$

[6 marks]

[TOTAL 25 marks]

4. (i) State Ampere's Law and explain the meaning of the various quantities and expressions involved. [5 marks]
- (ii) An infinite coaxial current system carries a current I distributed uniformly in a core of radius a . A cylindrical shell of radius $b > a$ carries an equal total current in the opposite direction as shown in the left figure.



- (a) Find the magnetic field in the region $a < r < b$ and show that the magnetic field is zero outside the system. [5 marks]
- (b) A short segment of length ℓ of such a configuration is used to deflect electrons (mass m , charge $-e$) initially travelling with $\mathbf{v} = v_o \hat{\mathbf{z}}$ at a radial position midway between the core and shell, i.e., at $r = (a + b)/2$. Assuming that $v_z \approx v_o$ throughout the subsequent motion, and neglecting any end effects due to the finite length of this segment, show that after traversing the length ℓ at a roughly constant radial distance $(a + b)/2$ each electron has received a positive radial impulse

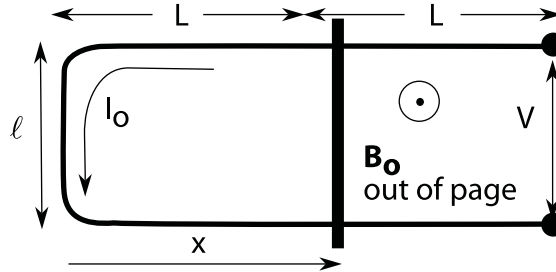
$$\Delta p_r = \int F_r dt \approx F_r \Delta t = \frac{\mu_o e I \ell}{\pi(a + b)},$$

where p_r is the radial momentum of the electron and F_r is the radial component of the force. [5 marks]

- (c) Use this result to determine the angular deflection δ of the electrons. [5 marks]
- (d) Hence estimate the current I required to use such a deflection system in a television requiring $\tan \delta = 0.1$ with the following parameters: $a = 1$ cm, $b = 2$ cm, $\ell = 4$ cm, $v_o = 2 \times 10^7$ m/s. [5 marks]

[TOTAL 25 marks]

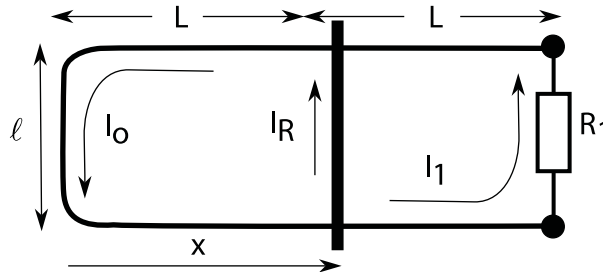
5. An induction slider is set up as shown in the following diagram. It has a U-shaped conductor of negligible resistance along which slides a conductor of resistance R . The assembly is immersed in a uniform magnetic field \mathbf{B} of magnitude B_0 that is directed out of the page. The slider is moved such that its position with respect to the end of the U-shaped conductor is given by $x = L + x_0 \cos \omega t$ with $x_0 \leq L$.



- (i) Find the voltage V appearing at the ends of the U and the current I_0 flowing around the U. [7 marks]
- (ii) If the U-shaped conductor has a finite resistance per unit length α , show that the voltage V appearing at the ends is now given by:

$$V = B_0 \ell x_0 \omega \sin \omega t \frac{R}{R + \alpha [\ell + 2(L + x_0 \cos \omega t)]}. \quad [6 \text{ marks}]$$

- (iii) A resistive load R_1 is now placed across the ends of the circuit, as shown in the following figure.



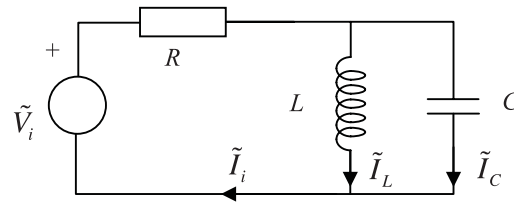
- (a) Show that $I_0 = I_R + I_1$. [3 marks]
- (b) By considering various closed loops within this circuit, derive any two of the following three relations:

$$\begin{aligned} I_R R + I_0 \alpha [\ell + 2(L + x_0 \cos \omega t)] &= B_0 \ell \omega x_0 \sin \omega t \\ -I_R R + I_1 [R_1 + 2\alpha(L - x_0 \cos \omega t)] &= -B_0 \ell \omega x_0 \sin \omega t \\ I_1 [R_1 + 2\alpha(L - x_0 \cos \omega t)] + I_0 \alpha [\ell + 2(L + x_0 \cos \omega t)] &= 0 \end{aligned} \quad [3 \text{ marks}]$$

- (c) Show that the limit $R_1 \rightarrow \infty$ must have $I_1 \rightarrow 0$ and that the first two of the above set of relations are then consistent with the results in part (ii) of this question. [3 marks]
- (d) For the case $\alpha \rightarrow 0$ show again that $I_1 \rightarrow 0$. [3 marks]

[TOTAL 25 marks]

6. The diagram below shows a parallel LC resonance circuit in series with a resistance, R , and driven by a voltage source $V_i(t)$.



- (i) By replacing the voltage source and resistor combination with a Norton equivalent, or otherwise, show that the current phasor in the inductor can be written as:

$$\tilde{I}_L = \frac{1}{1 + j\omega\frac{L}{R} - \omega^2 LC} \times \frac{\tilde{V}_i}{R},$$

and find a similar expression for the current phasor in the capacitor. [8 marks]

- (ii) By comparing the expression for current in the inductor with the form:

$$\tilde{I}_L = \frac{1}{1 + j\frac{\omega}{\omega_0 Q} - \frac{\omega^2}{\omega_0^2}} \times \frac{\tilde{V}_i}{R}.$$

Find expressions for the natural (resonant) angular frequency, ω_0 , and the quality factor, Q , and evaluate their values when the circuit elements are:

$$C = 100 \text{ nF}, \quad L = 10 \mu\text{H}, \quad R = 100 \Omega. \quad [7 \text{ marks}]$$

- (iii) The voltage source is set to drive the circuit sinusoidally at its natural frequency with an amplitude of 1V:

$$V_i(t) = 1.0 \cos(\omega_0 t) \text{ V}.$$

Calculate the current that flows in (a) the capacitor and (b) the inductor as functions of time and hence deduce the total current, $I_i(t)$, delivered by the voltage source to the circuit. [7 marks]

- (iv) How does the quality factor, Q , affect the behaviour of this circuit? [3 marks]

[TOTAL 25 marks]

End