UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2005

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

ELECTRICITY & MAGNETISM

For First-Year Physics Students

Monday 6th June 2005: 14.00 to 16.00

Answer ALL parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Three identical charges of 2×10^{-6} C are located at the corners of an equilateral (equal sided) triangle with sides of length 0.1 m. Calculate the force on each charge giving both the magnitude and the direction.

[6 marks]

(ii) State Ampere's law. A circular wire of radius R carries a current I. The current density is uniform throughout the wire. Derive expressions for the magnetic field B as a function of distance r from the centre of the wire for $0 < r < R$ and $r > R$. Sketch a graph of B against r for $0 < r < 3R$.

[7 marks]

(iii) A current I is carried by free electrons moving along a wire of cross-section A. The number density of free electrons is n and the time between collisions of each electron is τ. Derive an expression, based on the Drude model, for the electric field required to draw the current through the wire. If $I = 2$ Amp, $\tau = 10^{-15}$ s, $A = 10^{-6}\,\mathrm{m}^2$ and $n = 10^{29}\,\mathrm{m}^{-3}$, calculate

(a) the electric field and

(b) the energy gained by 0.1 m length of the wire in 1 minute by resistive (Ohmic) heating.

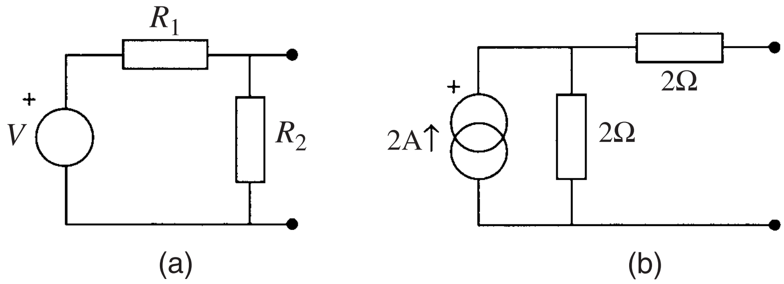
[8 marks]

(iv) An electric dipole (aligned along the x direction) consists of two opposite electric charges, $-Q$ and Q at $x = -h/2$ and $x = h/2$ respectively. Derive an exact expression for the potential ϕ at any point along the axis of the dipole at a distance x from the centre of the dipole when $x > h/2$. Given that the electric field on the axis is in the x direction, use your expression for ϕ to find an expression for the electric field on the axis for $x > h/2$.

[8 marks]

[TOTAL 29 marks]

2. (i) Find the Norton and Thévenin equivalents for the circuits shown below:

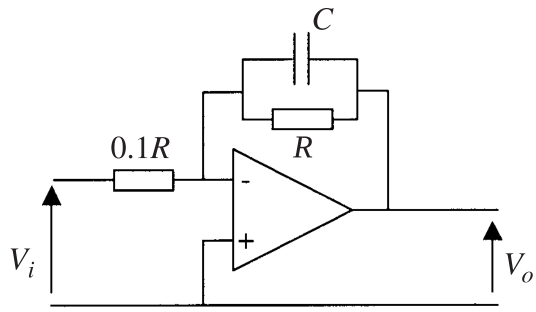


[6 marks]

(ii) State the virtual earth approximation as applied to circuits containing operational amplifiers.

By applying Kirchhoff's current law and summing currents at the inverting input of the op-amp in the circuit below, find the complex relationship between the phasor input and output voltages $\tilde{V}_o/\tilde{V}_i$ that is valid at all angular frequencies ω.

What function does this circuit perform?



[7 marks]

(iii) An $L = 1\,\text{mH}$ inductor is constructed by winding 100 turns of copper wire of diameter 0.2 mm around a cylindrical ferrite core of diameter 10 mm. Estimate the equivalent series resistance, R, of the constructed inductor if the copper has resistivity $\rho_{\text{Cu}} = 1.7 \times 10^{-8}\,\Omega\text{m}$.

Noting that the time constant of a series inductor-resistor combination is given by $\tau = L/R$, estimate the value of the time constant for this inductor.

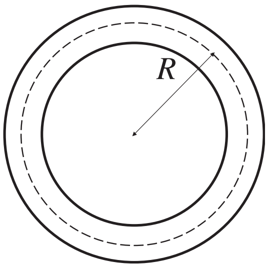
Without providing a full proof, sketch a graph of the time evolution of the current that would flow in this imperfect inductor when first connected across a perfect 1 V voltage source. Describe the significance of τ in terms of this graph.

[8 marks]

[TOTAL 21 marks]

SECTION B

3. (i)



A toroidal coil consists of N turns and carries a current I. Show that the magnetic field within the coil at a distance R from the centre of the coil is given by

$$B = \frac{\mu_0 N I}{2\pi R}.$$

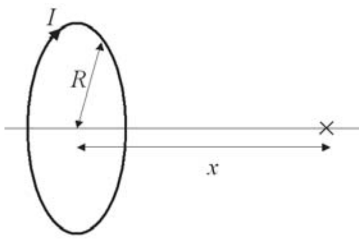
[6 marks]

(ii) Use this expression to find the magnetic field inside an infinitely long straight solenoid with n turns per unit length.

[4 marks]

(iii) The Biot-Savart law gives the contribution to the magnetic field made by a current element $I d\mathbf{l}$ in the form

$$d\mathbf{B} = \frac{\mu_0}{4\pi} I d\mathbf{l} \wedge \frac{\mathbf{r}}{r^3}.$$



Show that the magnitude of the magnetic field on the axis of a circular loop of radius R at a distance x from the centre (as shown above) is

$$B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}}$$

where I is the current flowing in the loop.

[8 marks]

(iv) By considering an infinite solenoid with n turns per unit length as a sum of circular coils with ndx turns in a small length dx, use this expression for B to find the magnetic field on the axis of the solenoid and show that it agrees with the expression derived in part (ii).

[7 marks]

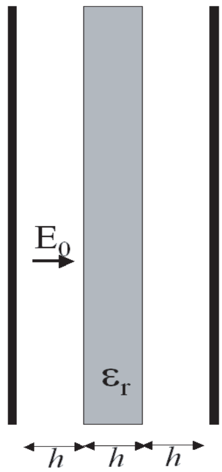
[TOTAL 25 marks]

[You may assume without proof that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(R^2 + x^2)^{3/2}} = \frac{2}{R^2}$.]

4. (i) State Gauss's flux law.

[3 marks]

(ii) An insulating slab with dielectric constant (relative permittivity) ε_r is placed in the centre of the space between the plates of a parallel plate capacitor as shown below.



The rest of the space is filled with air. The thickness of the slab is h and the distance between the plates is $3h$. Edge effects can be neglected. If the electric field is E_0 in the gaps between the dielectric slab and the capacitor plates, find expressions (in terms of E_0) for:

(a) the electric displacement D_d in the dielectric slab,

(b) the electric field E_d in the dielectric slab,

(c) the magnitude of the total charge density (per unit area) on each surface of the dielectric slab, and

(d) the magnitude of the charge density on the inner surfaces of the capacitor plates.

[12 marks]

(iii) Give one possible explanation of how a charge density forms at the surfaces of the dielectric slab.

[3 marks]

(iv) Derive an expression for the capacitance of this arrangement if the area of each surface of the slab and the capacitor plates is A.

[5 marks]

(v) Calculate the capacitance if $\varepsilon_r = 5$, $h = 3\,\text{mm}$ and $A = 4\,\text{cm}^2$.

[2 marks]

[TOTAL 25 marks]

5. (i) (a) If the velocity of a charged particle is initially perpendicular to the uniform magnetic field in which it moves, show that the component of its velocity parallel to the field remains zero. [4 marks]

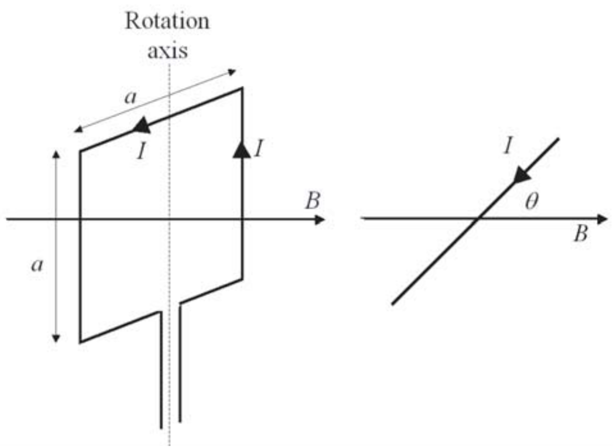
(b) A particle with charge q and mass m moves in the (x, y) plane in a magnetic field B which is in the z direction. Show that the x and y components of the particle's velocity obey the equations

$$\frac{d\mathrm{v}_x}{dt} = \frac{qB}{m}\mathrm{v}_y \qquad \frac{d\mathrm{v}_y}{dt} = -\frac{qB}{m}\mathrm{v}_x\,.$$

Describe the motion of the particle and the significance of the quantity qB/m. [6 marks]

(c) If electrons are projected across a magnetic field $\mathbf{B}$ with velocity $\mathbf{v}$ perpendicular to $\mathbf{B}$, show that an electric field $\mathbf{E}$ can be imposed which allows the electron to move in a straight line. Calculate the magnitude of $\mathbf{E}$ if the magnitude of the electron velocity $\mathbf{v}$ is $10^6\,\mathrm{m\,s^{-1}}$ and the magnitude of magnetic field $\mathbf{B}$ is 2 Tesla. Calculate the value of the scalar triple product $\mathbf{E}.(\mathbf{v} \wedge \mathbf{B})$. [6 marks]

(ii) A square loop of wire carrying a current I is placed in a magnetic field B and is allowed to rotate freely, without friction, about its vertical axis as shown in the left hand diagram. The length of each side is a.



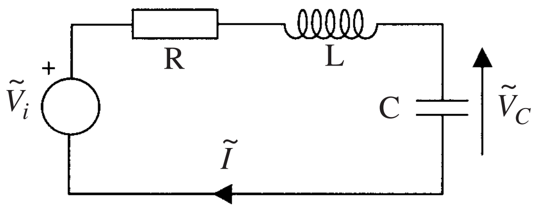
(a) The right hand diagram gives a vertical view from above the loop and defines the angle θ between the loop of wire and the magnetic field. Show that the torque on the loop about the axis has a magnitude $T = IBa^2\cos\theta$. [4 marks]

(b) If the loop is released from rest at $\theta = 80^\circ$, describe the subsequent motion of the loop assuming friction can be neglected.

In practice the rotation of the loop would be damped by friction. What would be the final resting position of the loop? [5 marks]

[TOTAL 25 marks]

6. The diagram below shows a series LCR resonant circuit driven by a sinusoidal voltage source $V_i(t)$ at an arbitrary angular frequency ω.



(i) State Kirchhoff's voltage law and use it to determine expressions for the complex phasor current $\tilde{I}$ and hence also for the phasor voltage, $\tilde{V}$, across the capacitor in terms of the circuit component values R, C and L and the input phasor voltage $\tilde{V}_i$. By writing your expressions in the form

$$\frac{(...)}{1 + j\dfrac{\omega}{\omega_o Q} - \dfrac{\omega^2}{\omega_o^2}}$$

determine the natural (resonant) frequency, ω_0, and the Q-factor, Q, for the circuit in terms of the component values.

[8 marks]

(ii) The circuit is driven at resonance by setting the input voltage $V_i(t) = V_i \cos(\omega_o t)$. Show that the current, $I(t)$, in the circuit as a function of time is given by:

$$I\ (t)\ \frac{V_i}{R} \cos\left(\omega_o t\right)$$

and hence find a similar expression for $V_c(t)$.

[6 marks]

(iii) Calculate the instantaneous energy stored in the capacitor, $U_C(t) =\hat{}\,C V_C^2(t)$, and the inductor, $U_L(t) =\hat{}\,L I^2(t)$, as functions of time and show that the total summed energy, U_T, stored in the inductor and capacitor is a constant given by:

$$U_T = U_C\ (t) + U_L\ (t) = \frac{V_i^2 L}{2R^2}\,.$$

[6 marks]

(iv) By considering only the RMS value of the current (and without performing an integral), obtain an expression for the time averaged power dissipated in the resistor $\langle P_R \rangle$ and hence show that:

$$\frac{\omega_o U_T}{\langle P_R \rangle} = Q\,.$$

[5 marks]

[TOTAL 25 marks]

End