Imperial College London
BSc/MSci EXAMINATION June 2008

This paper is also taken for the relevant Examination for the Associateship

ELECTRICITY & MAGNETISM

For First-Year Physics Students

Monday, 9th June 2008: 14:00 to 16:00

Answer both questions in Section A and TWO questions from Section B.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Three charges of +2 Coulombs each are placed at locations $(0,0)$m, $(0,1)$m, and $(1,0)$m in the $x-y$ plane. A charge of -2 Coulombs is placed at $(1,1)$m. Calculate the total force on this fourth charge. [5 marks]

(ii) (a) State Gauss's Flux Law explaining briefly what all the terms and symbols mean.

(b) A point charge Q is surrounded by a spherical distribution of charge of radius a that is uniform in charge density ρ for $0 < r < a$. Use Gauss's Law to find the electric field everywhere in space.

[8 marks]

(iii) An infinite cylindrical wire of radius a carries a uniform current density $\mathbf{j} = j_o\hat{\mathbf{z}}$.

(a) Find the total current carried by the wire.

(b) Indicate on a sketch the direction of the resulting magnetic field.

(c) Use Ampere's Law to find the magnitude of the magnetic field for $r < a$ and $r > a$.

[8 marks]

(iv) A square loop of side a lies in the $x-y$ plane. A magnetic field is present that is uniform in space but varies in time according to

$$\mathbf{B}(t) = B_o \cos \omega t\, \hat{\mathbf{z}}$$

(a) If the loop has a total electrical resistance R, find the current $I(t)$ flowing in the loop.

(b) Sketch the configuration of the loop and indicate clearly on your sketch the direction of I when $\omega t = \pi/2$.

[8 marks]

[Total 29 marks]

2. The open-loop voltage gain for an operational amplifier is known to be

$$A = \frac{10^5}{1 + j\omega/10}$$

(i) Express the low frequency gain in decibels. [1 mark]

(ii) Explain what is meant by the 'roll off' of the gain at high frequencies and state its value. [2 marks]

(iii) Make an appropriately labelled sketch of the Bode plot for the open-loop gain. [3 marks]

The amplifier is incorporated in the circuit shown below.

(iv) The circuit is a virtual earth amplifier. Explain what this means. [2 marks]

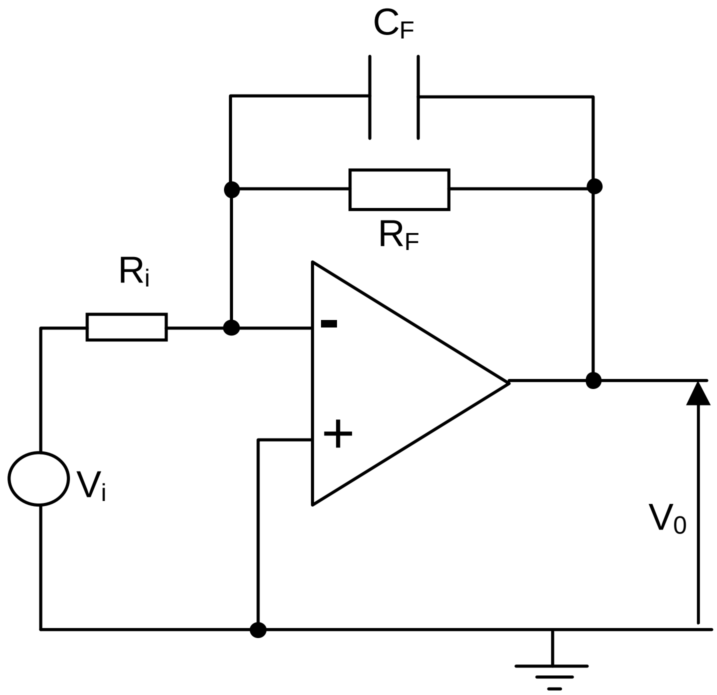
(v) Find an expression for the impedance of the parallel RC combination in the feed-back loop. [3 marks]

(vi) Using your answers to parts (iv) and (v) show that the closed loop gain is given by

$$G = -\frac{R_F/R_i}{1 + j\omega/\omega_0}$$

(where $\omega_o = 1/R_F C_F$). [5 marks]

(vii) Using values of $R_i = 1\,\mathrm{k\Omega}$, $R_F = 33\,\mathrm{k\Omega}$ and $C = 4.7\,\mathrm{pF}$ calculate the closed loop gain and the corner frequency and sketch the closed loop gain on the same Bode plot as for part (iii). [5 marks]



[Total 21 marks]

SECTION B

3. (i) An earthed infinite plane conductor occupies the $x-y$ plane. A charge $q > 0$ is placed along the z-axis at $z = d$.

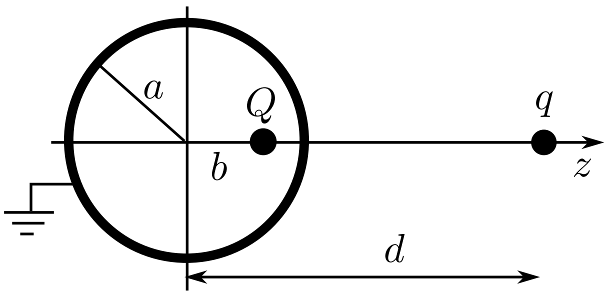
(a) Show that the appropriate image has a charge $-q$ and find its location, stating clearly your reasoning. [7 marks]

(b) Hence show that the electric potential in the region $z > 0$ is given by

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right) \quad \text{[3 marks]}$$

(c) Verify that $V = 0$ at the surface of the conductor. [3 marks]

(ii) Now, instead of the plane, an earthed spherical conductor of radius a is placed with its centre at the origin. A charge $q > 0$ is again placed at $z = d$ outside the sphere (i.e., $d > a$).



(a) Consider a possible image charge Q placed along the z-axis at $z = b$ inside the sphere, i.e., with $b < a$, as shown in the diagram above. Find an expression for the potential outside $|\mathbf{r}| = a$ due to q and Q. [5 marks]

(b) Show that for $V = 0$ at the surface of the sphere the charge Q must be negative and that

(1) $$q^2\left(a^2 - 2zb + b^2\right) = Q^2\left(a^2 - 2zd + d^2\right)$$

for all points on the sphere (i.e., for $-a \le z \le a$). [5 marks]

(c) Verify, e.g., by direct substitution, that an image charge with $Q = -qa/d$ located at $b = a^2/d$ satisfies the condition given in Equation (1). [2 marks]

[Total 25 marks]

4. (i) The Law of Biot and Savart relates a current I along an elemental path $\mathbf{dl}$ to the resulting magnetic field $\mathbf{B}$ at relative position $\mathbf{r}$ by

$$\mathbf{B} = \frac{\mu_o}{4\pi}\int \frac{I\mathbf{dl}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_o}{4\pi}\int \frac{I\mathbf{dl}\times\mathbf{r}}{r^3}$$

(a) A straight segment of wire of length $2a$ lies along the z-axis from $z=-a$ to $z=+a$. The wire carries a current I in the $+\hat{\mathbf{z}}$ direction. Show that the magnetic field due to this current at a point $P=(x_o,0,z_o)$ has magnitude

$$|\mathbf{B}| = \frac{\mu_o I x_o}{4\pi}\int_{-a-z_o}^{a-z_o} \frac{d\zeta}{(x_o^2+\zeta^2)^{3/2}}$$

where $\zeta = z - z_o$ and find the direction of $\mathbf{B}$. [8 marks]

(b) Evaluate this integral. [5 marks]

[You may assume without proof that $\displaystyle\int \frac{dx}{(b^2+x^2)^{3/2}} = \frac{x}{b^2\sqrt{b^2+x^2}}$]

(ii) A particle of mass m and charge q moves in the $x-y$ plane under the influence of the Lorentz force $q\mathbf{v}\times\mathbf{B}$. The magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$ is uniform.

(a) Show that the $\hat{\mathbf{x}}$ equation of motion can be written

$$\frac{dv_x}{dt} = \Omega v_y$$

and relate the constant Ω to the other parameters of the problem. [3 marks]

(b) Find an analogous equation of motion for v_y and hence show that

$$\frac{d}{dt}\left[\frac{1}{2}\left(v_x^2+v_y^2\right)\right] = v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt} = 0$$

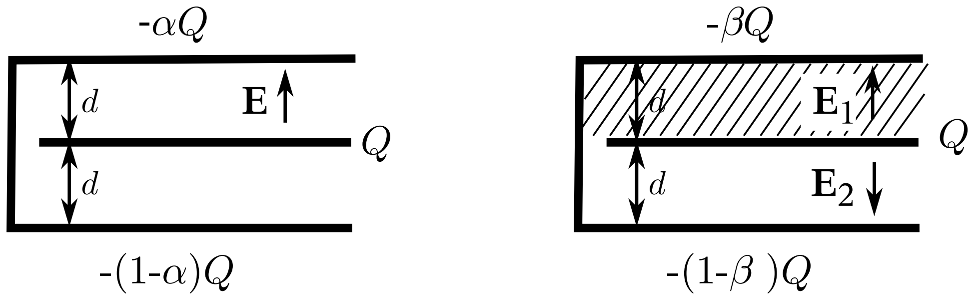
i.e., that $v^2 \equiv v_x^2+v_y^2$ is constant. [3 marks]

(c) Show further that the magnitude of the vector force on the particle is constant. [3 marks]

(d) Hence by equating this magnitude of the force to the centripetal force mv^2/R, find the radius R of the resulting circular motion. [3 marks]

[Total 25 marks]

5. (i) A capacitor is formed by folding a rectangular conducting sheet of area $2A$ around a sheet of area A, as shown in the following sketch. The gap between neighbouring surfaces is d. A charge of $+Q$ is placed on the central sheet while a charge $-Q$ is placed on the folded sheet and is divided into an unknown fraction α on the top segment and $(1-\alpha)$ on the bottom (left sketch).



(a) Neglecting any edge effects, determine α and find the capacitance of this system. [6 marks]

(b) A dielectric material with relative permittivity ε_r is now added and fills the upper gap region only, as sketched on the right; the lower portion is still filled only with air. Noting that the two segments must be at the same voltage, find the charge on the two segments of the folded sheet in this case, and show that the capacitance of this system is

$$C = \frac{\varepsilon_o(1+\varepsilon_r)A}{d}$$

[6 marks]

(ii) (a) A charge $+q$ is placed at the point $(0,0,a/2)$. An equal and opposite chage $-q$ is placed at $(0,0,-a/2)$. What is the vector dipole moment $\mathbf{p}$ of this configuration? [2 marks]

(b) Show that at large distances with respect to a the electrostatic potential of the dipole defined in (ii)(a) is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}$$

where $\mathbf{r} = (x,y,z)$ is the position vector from the origin and $r \equiv |\mathbf{r}|$. [You may use without proof $1/\sqrt{1+\delta} \approx 1 - \frac{1}{2}\delta$.] [6 marks]

(c) Show that the plane $z=0$ is an equipotential for this large-distance result. What does this tell you about the direction of the electric field for points in the $x-y$ plane? [5 marks]

[Total 25 marks]

6. (i) The Thévenin equivalent circuit of a source consists of a 3 kHz open circuit voltage of amplitude 5 V and source impedance, Z_S. The measured short circuit current has an amplitude of 5 mA and leads the voltage by 90°.

(a) By writing the voltage and current as phasors or complex quantities find the output impedance of the source. [3 marks]

(b) A 100 nF capacitor is now placed across the output terminals of the source. Calculate the amplitude and phase of the voltage across the capacitor. [5 marks]

(ii) For the circuit shown below:

(a) Derive an expression for the total resistance. [3 marks]

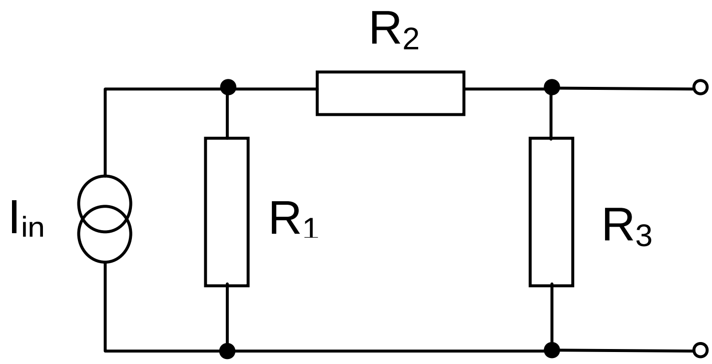
(b) Hence show that the open circuit voltage measured across the marked terminals is given by

$$V_{oc} = \frac{R_1 R_3}{R_1 + R_2 + R_3} I_{in}$$ [5 marks]

(c) Find an expression for the short circuit current I_{SC}. [4 marks]

(d) Using the results obtained in parts (b) and (c) find an expression for the source resistance, R_S. Hence sketch and label the Thévenin equivalent circuit. [3 marks]

(e) Sketch the Norton equivalent circuit and check it gives the correct output voltage. [2 marks]



[Total 25 marks]