First-Year Mathematics

Solutions to Problem Set 11 March 17, 2005

1. We solve the equation of motion of a classical undamped harmonic oscillator with natural frequency ω_0,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0\,, \tag{1}$$

with a trial solution $x(t) = e^{mt}$. Substituting this expression into the equation yields

$$m^2\, e^{mt} + \omega_0^2\, e^{mt} = (m^2 + \omega_0^2)\, e^{mt} = 0\,. \tag{2}$$

The characteristic equation is

$$m^2 + \omega_0^2 = (m - i\omega_0)(m + i\omega_0) = 0\,, \tag{3}$$

which has roots $m_1 = -i\omega_0$ and $m_2 = i\omega_0$. The general solution to the Eq. (1) is

$$x(t) = A\, e^{-i\omega_0 t} + B\, e^{i\omega_0 t}\,, \tag{4}$$

where A and B are determined by the initial conditions,

$$x(0) = x_0\,, \qquad \left.\frac{dx}{dt}\right|_{t=0} = x_0'\,. \tag{5}$$

Substitution of Eq. (4) into the initial conditions produces

$$x(0) = A + B = x_0\,, \tag{6}$$

$$\left.\frac{dx}{dt}\right|_{t=0} = -i\omega_0 A + i\omega_0 B = x_0'\,. \tag{7}$$

After dividing both sides of Eq. (7) by ω_0 and multiplying both sides by i, we obtain the two simultaneous equations for A and B in the form:

$$A + B = x_0\,, \tag{8}$$

$$A - B = \frac{ix_0'}{\omega_0}\,. \tag{9}$$

These equations are easily solved and we obtain

$$A = \frac{1}{2}\left(x_0 + \frac{ix_0'}{\omega_0}\right)\,, \tag{10}$$

$$B = \frac{1}{2}\left(x_0 - \frac{ix_0'}{\omega_0}\right)\,. \tag{11}$$

Thus, the solution to the initial-value problem is

$$\begin{aligned} x(t) &= \frac{1}{2}\left(x_0 + \frac{ix_0'}{\omega_0}\right) e^{-i\omega_0 t} + \frac{1}{2}\left(x_0 - \frac{ix_0'}{\omega_0}\right) e^{i\omega_0 t} \\ &= x_0\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) - \frac{ix_0'}{\omega_0}\left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2}\right) \\ &= x_0\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) + \frac{x_0'}{\omega_0}\left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}\right) \\ &= x_0 \cos\omega_0 t + \frac{x_0'}{\omega_0}\sin\omega_0 t\,. \end{aligned} \tag{12}$$

2. To obtain the general solution of

$$\frac{d^2y}{dx^2} + (E - V)\,y = 0\,, \tag{13}$$

we attempt a solution of the form $y(x) = \mathrm{e}^{mx}$ and choose m by the requirement that this expression is a solution. Substituting into Eq. (13) yields

$$(m^2 + E - V)\,\mathrm{e}^{mx} = 0\,. \tag{14}$$

The values of m are thus given by

$$m_1 = \sqrt{V - E}, \qquad m_2 = -\sqrt{V - E}\,. \tag{15}$$

Notice that if $V > E$, then m_1 and m_2 are *real*, which if $V < E$, the m_1 and m_2 are *imaginary* (and complex conjugates of one another). In either case, the general solution to Eq. (13) is

$$\psi(x) = A\,\mathrm{e}^{m_1 x} + B\,\mathrm{e}^{m_2 x}\,, \tag{16}$$

where A and B are constants to be determined by auxiliary conditions, of which there must be **two**. If m_1 and m_2 are real, then the general solution is combination of exponentially growing and decaying solutions, while if m_1 and m_2 are imaginary, the general solution is a combination of oscillating solutions.

3. For each of the differential equations given, two solutions may be found by first determining the roots of the characteristic equation and then following the procedure outlined in Section 8.3.

   (a) $y'' + 3y' + 2y = 0$. Comparing with the standard form (8.22) we identify

$$a = 1, \qquad b = 3, \qquad c = 2\,. \tag{17}$$

The roots of the characteristic equation are then given by

$$m = \tfrac{1}{2}\left(-3 \pm \sqrt{9-8}\,\right) = \tfrac{1}{2}(-3 \pm 1) = -1, -2\,. \tag{18}$$

This equation is of the type in Case I, so the two solutions are obtained as

$$y_1(x) = \mathrm{e}^{-x}, \qquad y_2(x) = \mathrm{e}^{-2x}\,. \tag{19}$$

(b) $y'' - 4y' + 5y = 0$. Comparing with the standard form (8.22) we identify

$$a = 1, \qquad b = -4, \qquad c = 5\,. \tag{20}$$

The roots of the characteristic equation are then given by

$$m = \tfrac{1}{2}\left(4 \pm \sqrt{16-20}\,\right) = \tfrac{1}{2}(4 \pm 4i) = 2 \pm 2i\,. \tag{21}$$

This equation is of the type in Case III, so the two solutions are obtained as

$$y_1(x) = \mathrm{e}^{2(1+i)x}, \qquad y_2(x) = \mathrm{e}^{2(1-i)x}\,. \tag{22}$$

(c) $y'' - 4y' + 4y = 0$. Comparing with the standard form (8.22) we identify

$$a = 1, \qquad b = -4, \qquad c = 4\,. \tag{23}$$

The roots of the characteristic equation are then given by

$$m = \tfrac{1}{2}\left(4 \pm \sqrt{16-16}\,\right) = 2\,. \tag{24}$$

This equation is of the type in Case II, so the two solutions are obtained as

$$y_1(x) = \mathrm{e}^{2x}, \qquad y_2(x) = x\,\mathrm{e}^{2x}\,. \tag{25}$$

4. Having the determined two solutions for each of the equations in Problem 1, we now form the general solution to fit to the initial conditions $y(0) = 1$, $y'(0) = -1$.

(a) $y'' + 3y' + 2y = 0$. The general solution is

$$y(x) = A\,\mathrm{e}^{-x} + B\,\mathrm{e}^{-2x}\,. \tag{26}$$

At $x = 0$, we have

$$y(0) = A + B = 1\,, \tag{27}$$

$$y'(0) = -A - 2B = -1\,, \tag{28}$$

which has the solution

$$A = 1, \qquad B = 0\,, \tag{29}$$

so the solution to the initial-value problem is

$$y(x) = \mathrm{e}^{-x}\,. \tag{30}$$

(b) $y'' - 4y' + 5y = 0$. The general solution is

$$y(x) = A\,\mathrm{e}^{2(1+i)x} + B\,\mathrm{e}^{2(1-i)x}\,. \tag{31}$$

At $x = 0$, we have

$$y(0) = A + B = 1\,, \tag{32}$$

$$y'(0) = 2(1+i)A + 2(1-i)B = -1\,, \tag{33}$$

which has the solution

$$A = \tfrac{1}{2} + \tfrac{3}{4}i, \qquad B = \tfrac{1}{2} - \tfrac{3}{4}i\,, \tag{34}$$

so the solution to the initial-value problem is

$$\begin{aligned} y(x) &= \left(\tfrac{1}{2} + \tfrac{3}{4}i\right)\mathrm{e}^{2(1+i)x} + \left(\tfrac{1}{2} - \tfrac{3}{4}i\right)\mathrm{e}^{2(1-i)x} \\ &= \mathrm{e}^{2x}\left[\cos(2x) - \tfrac{3}{2}\sin(2x)\right]. \end{aligned} \tag{35}$$

(c) $y'' - 4y' + 4y = 0$. The general solution is

$$y(x) = A\,\mathrm{e}^{2x} + Bx\,\mathrm{e}^{2x}\,. \tag{36}$$

At $x = 0$, we have

$$y(0) = A = 1\,, \tag{37}$$

$$y'(0) = 2A + B = -1\,, \tag{38}$$

which has the solution

$$A = 1, \qquad B = -3\,, \tag{39}$$

so the solution to the initial-value problem is

$$y(x) = \mathrm{e}^{2x} - 3x\,\mathrm{e}^{2x} = (1-3x)\,\mathrm{e}^{2x}\,. \tag{40}$$

5. Since this is a differential equation with constant coefficients, we attempt to solve this equation with a trial solution of the form $y(x) = e^{mx}$. Substituting this expression into the differential equation yields

$$m^4\, e^{mx} - e^{mx} = (m^4 - 1)\, e^{mx} = 0\,. \tag{41}$$

The characteristic equation is identified as

$$m^4 - 1 = 0\,, \tag{42}$$

which can be factored as

$$m^4 - 1 = (m^2 - 1)(m^2 + 1) = (m - 1)(m + 1)(m - i)(m + i) = 0\,, \tag{43}$$

so we obtain four distinct roots: $m = -1, 1, -i, i$. Accordingly, there are four solutions of the differential equation:

$$y_1(x) = e^{-x}\,, \quad y_2(x) = e^{x}\,, \quad y_3(x) = e^{-ix}\,, \quad y_4(x) = e^{ix}\,. \tag{44}$$

The general solution is a linear combination of these solutions:

$$y(x) = A\, e^{-x} + B\, e^{x} + C\, e^{-ix} + D\, e^{ix}\,, \tag{45}$$

where **four** initial conditions are required to determine the four constants A, B, C, and D.

6. To determine how the Euler equation behaves under the change of variable $x = e^t$ (or $t = \ln x$), we first need to determine how the derivatives are transformed. This is done by applying the chain rule:

$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} \tag{46}$$

$$= \frac{1}{x}\frac{d}{dt} = \mathrm{e}^{-t}\frac{d}{dt}\,, \tag{47}$$

$$\frac{d^2}{dx^2} = \frac{d^2t}{dx^2}\frac{d}{dt} + \left(\frac{dt}{dx}\right)^2\frac{d^2}{dt^2}$$

$$= -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x^2}\frac{d^2}{dt^2} = -\mathrm{e}^{-2t}\frac{d}{dt} + \mathrm{e}^{-2t}\frac{d^2}{dt^2} \tag{48}$$

Substituting these expressions into Euler's equation, yields

$$a\,\mathrm{e}^{2t}\left(-\mathrm{e}^{-2t}\frac{dy}{dt} + \mathrm{e}^{-2t}\frac{d^2y}{dt^2}\right) + b\,\mathrm{e}^{t}\left(\mathrm{e}^{-t}\frac{dy}{dt}\right) + cy$$

$$= a\frac{d^2y}{dt^2} + (b - a)\frac{dy}{dt} + cy = 0\,, \tag{49}$$

which is a second-order equation with constant coefficients. We consider the three cases discussed in Section 8.3.

Case I. There are two real roots m_1 and m_2 of the characteristic equation, which yield the two solutions

$$y_1(t) = \mathrm{e}^{m_1 t}, \qquad y_2(t) = \mathrm{e}^{m_2 t}\,. \tag{50}$$

Case II. There is one real root, m_1, from which we obtain the two solutions

$$y_1(t) = \mathrm{e}^{m_1 t}, \qquad y_2(t) = t\,\mathrm{e}^{m_1 t}\,. \tag{51}$$

Case III. There are two roots, m_1 and m_1^*, which are complex conjugates, and we obtain the two solutions

$$y_1(t) = \mathrm{e}^{m_1 t}, \qquad y_2(t) = \mathrm{e}^{m_1^* t}\,. \tag{52}$$

To express these solutions in terms of the original variables, we substitute the relation $t = \ln x$ into each of the solutions in Eqs. (50), (51), and (52):

$$y_1(x) = x^{m_1}\,, \qquad y_2(x) = x^{m_2}\,, \qquad \text{(Case I)} \tag{53}$$

$$y_1(x) = x^{m_2}\,, \qquad y_2(x) = x^{m_1}\ln x\,, \qquad \text{(Case II)} \tag{54}$$

$$y_1(x) = x^{m_1}\,, \qquad y_2(x) = x^{m_1^*}\,. \qquad \text{(Case III)} \tag{55}$$