First-Year Mathematics

Problem Set 11 March 17, 2005

1. Consider the equation of motion of a classical undamped harmonic oscillator with natural frequency ω_0,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0\,, \tag{1}$$

with the initial conditions

$$x(0) = x_0\,, \qquad \left.\frac{dx}{dt}\right|_{t=0} = x_0'\,. \tag{2}$$

Show that the solution to this initial value problem is

$$x(t) = x_0 \cos(\omega_0 t) + \frac{x_0'}{\omega_0}\sin(\omega_0 t)\,. \tag{3}$$

2. The Schrödinger equation that describes the stationary states of a particle of mass m in a one-dimensional potential $V(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

where $\hbar = h/2\pi$, E is the energy of the particle and ψ is the wavefunction. Determine the general solution to this equation for the case where the potential is independent of position, i.e. where $V(x) = V$ is a constant. How many auxiliary conditions are required to determine the arbitrary constants in this solution?

3. Find the general solution of each of the following equations:

(a) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$

(b) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = 0$

(c) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$

Ans: (a) $y(x) = A\,\mathrm{e}^{-x} + B\,\mathrm{e}^{-2x}$, (b) $y(x) = A\,\mathrm{e}^{(2+i)x} + B\,\mathrm{e}^{(2-i)x}$, (c) $y(x) = A\,\mathrm{e}^{2x} + B\,x\,\mathrm{e}^{2x}$.

4. Find the solutions to each of the equations in Problem 1 for the initial conditions: $y(0) = 1$ and $y'(0) = -1$.

   Ans: (a) $y(x) = \mathrm{e}^{-x}$, (b) $y(x) = \mathrm{e}^{2x}[\cos x - 3\sin x]$, (c) $y(x) = (1 - 3x)\,\mathrm{e}^{2x}$.

5. According to the theory of beam bending, the deflection y of a beam from the horizontal is determined by a fourth-order differential equation of the form

$$\frac{d^4y}{dx^4} - y = 0\,.$$

   Determine the general solution of this equation. How many auxiliary conditions are required to determine the arbitrary constants in the general solution?

   Ans: $y(x) = A\,\mathrm{e}^{-x} + B\,\mathrm{e}^{x} + C\,\mathrm{e}^{-ix} + D\,\mathrm{e}^{ix}$; four conditions are required to determine the constants A, B, C, D.

6.* Consider the following equation:

$$a\,x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\,x\frac{\mathrm{d}y}{\mathrm{d}x} + c\,y = 0$$

   where a, b and c are constants. This is known as **Euler's differential equation**. By changing the independent variable from x to t according to $x=\mathrm{e}^t$, show that this equation becomes

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (b - a)\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0$$

   Beginning with two solutions of this equation for each of the three cases discussed in Section 1.3 (you do not need to obtain the roots of the characteristic equation explicitly), determine the corresponding solutions of Euler's equation as a function of x.