First-Year Mathematics

Solutions to Problem Set 10 March 11, 2005

1. (a) The bounding curve for the paraboloid is the circle $x^2+y^2=R^2$ in the x–y plane. This curve and the stated vector field are the same as those discussed in lectures (for the upper half-sphere), where it was found that

$$\oint_{\partial\sigma} \boldsymbol{V}\cdot d\boldsymbol{r} = 2\pi R^2\,. \tag{1}$$

Thus, according to Stokes' theorem, the integral

$$\iint_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma\,, \tag{2}$$

where σ is the paraboloid for $z\ge 0$, also has this value because the two surfaces have the same bounding curve.

(b) We have that $\boldsymbol{V}\cdot d\boldsymbol{r} = -y\,dx + x\,dy$ for any path in x–y plane. The integral over a closed curve in the x–y plane is therefore

$$\oint_{\partial\sigma} (x\,dy - y\,dx) \tag{3}$$

In circular polar coordinates,

$$x = R\cos\phi\,, \qquad y = R\sin\phi\,, \tag{4}$$

with $0\le\phi\le 2\pi$, we have

$$dx = -R\sin\phi\,d\phi\,, \qquad dy = R\cos\phi\,d\phi\,. \tag{5}$$

The loop integral is therefore given by

$$\oint_{\partial\sigma} (x\,dy - y\,dx) = R^2\int_0^{2\pi} d\phi = 2\pi R^2\,. \tag{6}$$

(c) The curl of $\boldsymbol{V}$ is $\boldsymbol{\nabla}\times\boldsymbol{V} = 2\,\boldsymbol{k}$. The surface unit normal was found in lectures to be

$$\boldsymbol{n} = \frac{x}{R}\,\boldsymbol{i} + \frac{y}{R}\,\boldsymbol{j} + \frac{z}{R}\,\boldsymbol{k}\,, \tag{7}$$

so $(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n} = 2z/R$. As in the lectures, the surface integral is calculated in spherical polar coordinates:

$$d\sigma = R^2\sin\theta\,d\theta\,d\phi, \qquad z = R\cos\theta\,, \tag{8}$$

where $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \frac{1}{2}\pi$. The integral is

$$\iint_\sigma (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n}\, d\sigma = 2R^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta\cos\theta = 2\pi R^2 \sin^2\theta \Big|_0^{\pi/2} = 2\pi R^2\,, \tag{9}$$

which agrees with the result of Part (b).

(d) The corresponding calculation for the lower half-sphere is carried out by observing that the limits on the θ-integration are now given by $\frac{1}{2}\pi \leq \theta \leq \pi$, so the surface integral is

$$\iint_\sigma (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n}\, d\sigma = 2R^2 \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} d\theta \sin\theta\cos\theta = 2\pi R^2 \sin^2\theta \Big|_{\pi/2}^{\pi} = -2\pi R^2\,, \tag{10}$$

which is the *negative* of the result obtained for the *upper* half-sphere. This is due ultimately to the "right-hand rule" for the orientation between the curl and the unit normal.

(e) From the results in Parts (c) and (d) for the upper and lower half-spheres, we deduce that

$$\iint_\sigma (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n}\, d\sigma = 0\,, \tag{11}$$

where σ is the entire surface of the sphere of radius R. Indeed, this result can be generalized to any closed surface. Consider a cut through a plane parallel to the x–y plane which divides the surface into upper and lower parts. According to Stokes' theorem and the results of Parts (c) and (d) the value of

$$\iint_\sigma (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n}\, d\sigma \tag{12}$$

for the upper and lower parts of the surface have the same *absolute value*, but *opposite signs*. Thus, the integral over the entire surface vanishes.

2. (a) The stationary solutions of the logistic equation are determined by solving $dN/dt = 0$:

$$\left(1 - \frac{N}{\beta}\right) N = 0\,, \tag{13}$$

which yields the two solutions

$$N = 0 \qquad \text{and} \qquad N = \beta\,. \tag{14}$$

(b) The solution of the logistic equation proceeds by observing that it is a separable equation, so we first write:

$$\frac{dN}{\alpha\left(1-\dfrac{N}{\beta}\right)N} = \alpha dt\,. \tag{15}$$

Integration of the left-hand side requires partial fractions:

$$\frac{1}{\left(1-\dfrac{N}{\beta}\right)N} = \frac{A}{N} + \frac{B}{1-\dfrac{N}{\beta}}\,, \tag{16}$$

or

$$A\left(1-\frac{N}{\beta}\right) + BN = 1\,. \tag{17}$$

Choosing in turn $N = 0$ and $N = \beta$, yields $A = 1$ and $B = 1/\beta$. Thus,

$$\frac{dN}{N} + \frac{dN}{\beta - N} = \alpha dt\,. \tag{18}$$

By integrating from $N = N_0$ at $t = 0$ to $N = N(t)$ at t, we obtain

$$\ln\left[\frac{N(t)}{N_0}\right] - \ln\left[\frac{\beta - N(t)}{\beta - N_0}\right] = \alpha t\,. \tag{19}$$

Solving for $N(t)$ yields

$$N(t) = \frac{N_0\beta}{N_0 + (\beta - N_0)\,\mathrm{e}^{-\alpha t}}\,. \tag{20}$$

(c) For $0 < N_0 < \beta$,

$$\lim_{t\to\infty} N(t) = \beta \tag{21}$$

(d) The asymptotic solution $N = 0$ is obtained only if $N_0 = 0$. In particular if a solution starts *near* zero, it eventually reaches the solution $N = \beta$.

3. (a) The daily rate of change of the population P is given by

$$\frac{dP}{dt} = rP + 15 - 16 - 7 = rP - 8\,. \tag{22}$$

Initially, there are 100 insects, so

$$P(0) = 100\,. \tag{23}$$

(b) By introducing the quantity Q, defined in terms of P by

$$P = Q + \frac{8}{r}\,, \tag{24}$$

the differential equation for Q is

$$\frac{dP}{dt} = \frac{dQ}{dt} = r\left(Q + \frac{8}{r}\right) - 8 = rQ\,, \tag{25}$$

i.e.

$$\frac{dQ}{dt} = rQ\,. \tag{26}$$

The initial condition for Q is determined from

$$P(0) = Q(0) + \frac{8}{r} = 100\,, \tag{27}$$

which yields

$$Q(0) = 100 - \frac{8}{r}\,. \tag{28}$$

This equation has the same form as that solved in lectures. The solution is given by

$$Q(t) = Q(0)\,e^{rt} = \left(100 - \frac{8}{r}\right)\,e^{rt}\,. \tag{29}$$

The solution P of the original equation is therefore given by

$$\begin{aligned} P(t) &= Q(t) + \frac{8}{r} = \left(100 - \frac{8}{r}\right)\,e^{rt} + \frac{8}{r} \\ &= 100\,e^{rt} + \frac{8}{r}\left(1 - e^{rt}\right)\,. \end{aligned} \tag{30}$$

(c) The observation that, in the absence of outside factors, the population triples in two weeks can be used to determine r. With no outside factors, the differential equation for P is

$$\frac{dP}{dt} = rP\,. \tag{31}$$

with P(0)=100 . The solution is, in this case, given by

$$P(t) = 100\,e^{rt}\,. \tag{32}$$

Thus, the condition for r is

$$P(14) = 100\,e^{14r} = 300\,, \tag{33}$$

which yields

$$r = \frac{1}{14}\ln 3\,. \tag{34}$$

(d) The complete solution $P(t)$ is given by

$$P(t) = 100 \exp\left(\frac{\ln 3}{14}t\right) + \frac{112}{\ln 3}\left[1 - \exp\left(\frac{\ln 3}{14}t\right)\right] . \tag{35}$$

Setting $P = 0$, yields

$$\frac{112}{\ln 3} = \left(\frac{112}{\ln 3} - 100\right) \exp\left(\frac{\ln 3}{14}t\right) . \tag{36}$$

or,

$$\exp\left(\frac{\ln 3}{14}t\right) = \frac{112}{112 - 100 \ln 3} . \tag{37}$$

Thus, the time t^* at which the population vanishes is

$$t^* = \frac{14}{\ln 3} \ln\left(\frac{112}{112 - 100 \ln 3}\right) = 50.44 \,\text{days} . \tag{38}$$

Thus, the population of insects ceases to exist after this time. The solution is plotted below: