First-Year Mathematics

Problem Set 10 March 11, 2005

1. Stokes' theorem states that

$$\oint_{\partial\sigma} \boldsymbol{V}\cdot d\boldsymbol{r} = \iint_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma\,,$$

where $\boldsymbol{V}$ is a vector field and $\partial\sigma$ is the bounding curve for the surface σ.

(a) Given the vector field $\boldsymbol{V} = -y\,\boldsymbol{i} + x\,\boldsymbol{j} + z\,\boldsymbol{k}$, use Stokes' theorem to determine value of the integral

$$\iint_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma$$

over the paraboloid

$$x^2 + y^2 + z = R^2$$

for $z \geq 0$.

Hint: You do not need to evaluate this integral directly!

(b) Evaluate

$$\iint_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma$$

over *any* surface with $z \geq 0$ whose bounding curve is the circle $x^2 + y^2 = R^2$ for $\boldsymbol{V}$ given in Part (a).

(c) Evaluate *directly* the integral

$$\iint_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma$$

over the upper half-sphere of radius R,

$$x^2 + y^2 + z^2 = R^2$$

with $z \geq 0$ for $\boldsymbol{V}$ given in Part (a).

(d) Calculate the corresponding integral in (c) over the *lower* half-sphere ($z \leq 0$).

(e) Evaluate the integral

$$\iint_\sigma (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot\boldsymbol{n}\,d\sigma$$

over the *entire* surface of the sphere of radius R for $\boldsymbol{V}$ given in Part (a).

2. The **logistic equation**,

$$\frac{dN}{dt} = \alpha\left(1 - \frac{N}{\beta}\right)N\,,$$

where α and β are constants, is used to describe the growth of a population of species N mediated by the population itself to prevent unrestricted growth.

(a) What are the stationary solutions of this equation, i.e., those for which $dN/dt = 0$?

(b) Integrate this equation with the initial condition $N(0) = N_0$ to obtain the solution

$$N(t) = \frac{\beta N_0}{N_0 + (\beta - N_0)\,\mathrm{e}^{-\alpha t}}\,.$$

(c) What is the asymptotic solution ($t \to \infty$) if $0 < N_0 < \beta$?

(d) For what what conditions is the asymptotic solution $N = 0$? Suppose a solution starts near zero. As t increases, what value does N approach?

3. A population P of insects in a region grow at a rate r proportional to their current population. In the absence of any outside factors the population will triple in two weeks. On any given day there is a net migration into the region of 15 insects, 16 are eaten by the local bird population, and 7 die of natural causes. If there are initially 100 insects in the region, will the population survive? Proceed as follows:

(a) Show that the differential equation for the population P is

$$\frac{dP}{dt} = rP - 8\,,$$

with the initial condition

$$P(0) = 100\,.$$

(b) This equation can be solved by introducing a function Q related to P by $P = Q + 8/r$. Show that the initial-value problem for Q is

$$\frac{dQ}{dt} = rQ\,, \qquad Q(0) = 100 - \frac{8}{r}\,.$$

Solve this equation and thereby determine $P(t)$.

(c) The observation that, in the absence of outside factors, the population triples in two weeks can be used to determine r. Without outside factors, show that the solution for P is $P(t) = 100\,e^{rt}$. Hence, the condition for r is

$$100\,e^{14r} = 300\,.$$

(d) Determine whether or not the solution remains positive for all time.